

# Affine Mirković-Vilonen polytopes

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## Abstract

Each integrable lowest weight representation of a symmetrizable Kac-Moody Lie algebra  $\mathfrak{g}$  has a crystal in the sense of Kashiwara, which describes its combinatorial properties. For a given  $\mathfrak{g}$ , there is a limit crystal, usually denoted by  $B(-\infty)$ , which contains all the other crystals. When  $\mathfrak{g}$  is finite dimensional, a convex polytope, called the Mirković-Vilonen polytope, can be associated to each element in  $B(-\infty)$ . This polytope sits in the dual space of a Cartan subalgebra of  $\mathfrak{g}$ , and its edges are parallel to the roots of  $\mathfrak{g}$ . In this paper, we generalize this construction to the case where  $\mathfrak{g}$  is a symmetric affine Kac-Moody algebra. The datum of the polytope must however be complemented by partitions attached to the edges parallel to the imaginary root  $\delta$ . We prove that these decorated polytopes are characterized by conditions on their normal fans and on their 2-faces. In addition, we discuss how our polytopes provide an analog of the notion of Lusztig datum for affine Kac-Moody algebras. Our main tool is an algebro-geometric model for  $B(-\infty)$  constructed by Lusztig and by Kashiwara and Saito, based on representations of the completed preprojective algebra  $\Lambda$  of the same type as  $\mathfrak{g}$ . The underlying polytopes in our construction are described with the help of Buan, Iyama, Reiten and Scott's tilting theory for the category  $\Lambda\text{-mod}$ . The partitions we need come from studying the category of semistable  $\Lambda$ -modules of dimension-vector a multiple of  $\delta$ .

## 1 Introduction

Let  $A$  be a symmetrizable generalized Cartan matrix, with rows and columns indexed by a set  $I$ . We denote by  $\mathfrak{g}$  the Kac-Moody algebra defined by  $A$ . It comes with a triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ , with a root system  $\Phi$ , and with a Weyl group  $W$ . The simple roots  $\alpha_i$  are indexed by  $I$  and the group  $W$  is a Coxeter system, generated by the simple reflections  $s_i$ . We denote the length function of  $W$  by  $\ell : W \rightarrow \mathbb{N}$  and the set of positive (respectively, negative) roots by  $\Phi_+$  (respectively,  $\Phi_-$ ). The root lattice is denoted by  $\mathbb{Z}I = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$  and we set  $\mathbb{R}I = \mathbb{Z}I \otimes_{\mathbb{Z}} \mathbb{R}$ . Finally, we denote by  $\mathbb{R}_{\geq 0}I$  the set of linear combinations of the simple roots with nonnegative coefficients and we set  $\mathbb{N}I = \mathbb{Z}I \cap \mathbb{R}_{\geq 0}I$ .

## 1.1 Crystals

The combinatorics of the representation theory of  $\mathfrak{g}$  is captured by Kashiwara's theory of crystals. Let us summarize quickly this theory; we refer the reader to the nice survey [33] for detailed explanations.

A  $\mathfrak{g}$ -crystal is a set  $B$  endowed with maps  $\text{wt}$ ,  $\varepsilon_i$ ,  $\varphi_i$ ,  $\tilde{e}_i$  and  $\tilde{f}_i$ , for each  $i \in I$ , that satisfy certain axioms. This definition is of combinatorial nature and the axioms stipulate the local behavior of the structure maps around an element  $b \in B$ . This definition is however quite permissive, so one wants to restrict to crystals that actually come from representations.

In this respect, an important object is the crystal  $B(-\infty)$ , which contains the crystals of all the irreducible lowest weight integrable representations of  $\mathfrak{g}$  (see Theorem 8.1 in [33]). This crystal contains a lowest weight element  $u_{-\infty} \in B(-\infty)$  annihilated by all the lowering operators  $\tilde{f}_i$ , and any element of  $B(-\infty)$  can be obtained by applying a sequence of raising operators  $\tilde{e}_i$  to  $u_{-\infty}$ .

The crystal  $B(-\infty)$  itself is defined as a basis of the quantum group  $U_q(\mathfrak{n}_+)$  in the limit  $q \rightarrow 0$ . Working with this algebraic construction is cumbersome, and there exist other, more handy, algebro-geometric or combinatorial models for  $B(-\infty)$ .

One of these combinatorial models is Mirković-Vilonen (MV) polytopes. In this model, proposed by Anderson [2], one associates a convex polytope  $\text{Pol}(b) \subseteq \mathbb{R}I$  to each element  $b \in B(-\infty)$ . The construction of  $\text{Pol}(b)$  is based on the geometric Satake correspondence. More precisely, the affine Grassmannian of the Langlands dual of  $\mathfrak{g}$  contains remarkable subvarieties, called MV cycles after Mirković and Vilonen [43]. There is a natural bijection  $b \mapsto Z_b$  from  $B(-\infty)$  onto the set of all MV cycles [10, 11, 21], and  $\text{Pol}(b)$  is simply the image of  $Z_b$  by the moment map.

Using Berenstein and Zelevinsky's work [8], the second author showed in [32] that these MV polytopes can be described in a completely combinatorial fashion: these are the convex lattice polytopes, whose normal fan is a coarsening of the Weyl fan in the dual of  $\mathbb{R}I$ , and whose 2-faces have a shape constrained by the tropical Plücker relations. In addition, the length of the edges of  $\text{Pol}(b)$  is given by the Lusztig data of  $b$ , which indicate how  $b$ , viewed as a basis element of  $U_q(\mathfrak{g})$  at the limit  $q \rightarrow 0$ , compares with the PBW bases.

Until now, MV polytopes only existed for finite dimensional  $\mathfrak{g}$ . This paper aims at generalizing this model of MV polytopes to the case where  $\mathfrak{g}$  is an affine Kac-Moody algebra.

## 1.2 The preprojective model

Obstacles pop up when one tries to generalize the above constructions of  $\text{Pol}(b)$  to the affine case. Despite difficulties in defining the double-affine Grassmannian, the algebro-geometric model of  $B(-\infty)$  using MV cycles still exists in the affine case, thanks to Braverman, Finkelberg and Gaitsgory's work [10]; however, there is no obvious way to go from MV cycles to MV polytopes. (There is some recent work in this direction due to Muthiah [45].) On the combinatorial side, several PBW bases for  $U_q(\mathfrak{n}_+)$  have been defined in the affine case by Beck [6] and Ito [27], but the relationship between the different Lusztig data they provide has not been studied.

We are thus led to use a third construction of  $\text{Pol}(b)$ , recently obtained by the first two authors for the case of a finite dimensional  $\mathfrak{g}$  [5]. This construction uses a geometric model for  $B(-\infty)$  based on quiver varieties, which we now recall.

This model exists for any Kac-Moody algebra  $\mathfrak{g}$  (not necessarily of finite or affine type) but only when the generalized Cartan matrix  $A$  is symmetric. Then  $2\text{id} - A$  is the incidence matrix of the Dynkin graph  $(I, E)$ ; here our index set  $I$  serves as the set of vertices and  $E$  is the set of edges. Choosing an orientation of this graph yields a quiver  $Q$ , and one can then define the completed preprojective algebra  $\Lambda$  of  $Q$ .

A  $\Lambda$ -module is an  $I$ -graded vector space equipped with linear maps. If the dimension-vector is given, we can work with a fixed vector space; the datum of a  $\Lambda$ -module then amounts to the family of linear maps, which can be regarded as a point of an algebraic variety. This variety is called Lusztig's nilpotent variety; we denote it by  $\Lambda(\nu)$ , where  $\nu \in \mathbb{N}I$  is the dimension-vector. Abusing slightly the language, we often view a point  $T \in \Lambda(\nu)$  as a  $\Lambda$ -module.

For  $\nu \in \mathbb{N}I$ , let  $\mathfrak{B}(\nu)$  be the set of irreducible components of  $\Lambda(\nu)$ . We set  $\mathfrak{B} = \bigsqcup_{\nu \in \mathbb{N}I} \mathfrak{B}(\nu)$ . In [38], Lusztig endows  $\mathfrak{B}$  with the structure of a crystal, and in [34], Kashiwara and Saito show the existence of an isomorphism of crystals  $b \mapsto \Lambda_b$  from  $B(-\infty)$  onto  $\mathfrak{B}$ . This isomorphism is unique since  $B(-\infty)$  has no non-trivial automorphisms.

Given a finite-dimensional  $\Lambda$ -module  $T$ , we can consider the dimension-vectors of the  $\Lambda$ -submodules of  $T$ ; they are finitely many, since they belong to a bounded subset of the lattice  $\mathbb{Z}I$ . The convex hull in  $\mathbb{R}I$  of these dimension-vectors will be called the Harder-Narasimhan (HN) polytope of  $T$  and will be denoted by  $\text{Pol}(T)$ .

The main result of [5] is equivalent to the following statement: if  $\mathfrak{g}$  is finite dimensional, then for each  $b \in B(-\infty)$ , the set  $\{T \in \Lambda_b \mid \text{Pol}(T) = \text{Pol}(b)\}$  contains a dense open subset of  $\Lambda_b$ . In other words,  $\text{Pol}(b)$  is the general value of the map  $T \mapsto \text{Pol}(T)$  on  $\Lambda_b$ .

This result obviously suggests a general definition for MV polytopes. We will however see that for  $\mathfrak{g}$  of affine type, another piece of information is needed to have a complete model for

$B(-\infty)$ ; namely, we need to equip each polytope with a family of partitions. Our task now is to explain what our polytopes look like, and where these partitions come from.

### 1.3 Faces of HN polytopes

Choose a linear form  $\theta : \mathbb{R}I \rightarrow \mathbb{R}$  and let  $\psi_{\text{Pol}(T)}(\theta)$  denote the maximum value of  $\theta$  on  $\text{Pol}(T)$ . Then  $P_\theta = \{x \in \text{Pol}(T) \mid \langle \theta, x \rangle = \psi_{\text{Pol}(T)}(\theta)\}$  is a face of  $\text{Pol}(T)$ . Moreover, the set of submodules  $X \subseteq T$  whose dimension-vectors belong to  $P_\theta$  has a smallest element  $T_\theta^{\min}$  and a largest element  $T_\theta^{\max}$ .

The existence of  $T_\theta^{\min}$  and  $T_\theta^{\max}$  follows from general considerations: if we define the slope of a finite dimensional  $\Lambda$ -module  $X$  as  $\langle \theta, \underline{\dim} X \rangle / \dim X$ , then  $T_\theta^{\max}/T_\theta^{\min}$  is the semistable subquotient of slope zero in the Harder-Narasimhan filtration of  $T$ . Introducing the (abelian) subcategory  $\mathcal{R}_\theta$  of semistable  $\Lambda$ -modules of slope zero it follows that, for each submodule  $X \subseteq T$ ,

$$\underline{\dim}(X) \in P_\theta \iff \left( T_\theta^{\min} \subseteq X \subseteq T_\theta^{\max} \quad \text{and} \quad X/T_\theta^{\min} \in \mathcal{R}_\theta \right).$$

In other words, the face  $P_\theta$  coincides with the HN polytope of  $T_\theta^{\max}/T_\theta^{\min}$ , computed relative to the category  $\mathcal{R}_\theta$ , and shifted by  $\underline{\dim} T_\theta^{\min}$ .

Our aim now is to describe the normal fan to  $\text{Pol}(T)$ , that is, to understand how  $T_\theta^{\min}$ ,  $T_\theta^{\max}$  and  $\mathcal{R}_\theta$  depend on  $\theta$ . For that, we need tools that are specific to preprojective algebras.

### 1.4 Tits cone and tilting theory

One of these tools is Buan, Iyama, Reiten and Scott's tilting ideals for  $\Lambda$  [13]. Let  $S_i$  be the simple  $\Lambda$ -module of dimension-vector  $\alpha_i$  and let  $I_i$  be its annihilator, a one-dimensional two-sided ideal of  $\Lambda$ . The products of these ideals  $I_i$  are known to satisfy the braid relations, so to each  $w$  in the Weyl group of  $\mathfrak{g}$ , we can attach a two-sided ideal  $I_w$  of  $\Lambda$  by the rule  $I_w = I_{i_1} \cdots I_{i_\ell}$ , where  $s_{i_1} \cdots s_{i_\ell}$  is any reduced decomposition of  $w$ . Given a finite-dimensional  $\Lambda$ -module  $T$ , we denote the image of the evaluation map  $I_w \otimes_\Lambda \text{Hom}_\Lambda(I_w, T) \rightarrow T$  by  $T^w$ .

Recall that the dominant Weyl chamber  $C_0$  and the Tits cone  $C_T$  are the convex cones in the dual of  $\mathbb{R}I$  defined as

$$C_0 = \{\theta : \mathbb{R}I \rightarrow \mathbb{R} \mid \forall i \in I, \theta(\alpha_i) > 0\} \quad \text{and} \quad C_T = \bigcup_{w \in W} w \overline{C_0}.$$

We will show the equality  $T_\theta^{\min} = T_\theta^{\max} = T^w$  for any  $\Lambda$ -module  $T$ , any  $w \in W$  and any linear form  $\theta \in wC_0$ . This implies that  $\underline{\dim} T^w$  is a vertex of  $\text{Pol}(T)$  and that the normal cone to

$\text{Pol}(T)$  at this vertex contains  $wC_0$ . This also implies that  $\text{Pol}(T)$  is contained in

$$\{x \in \mathbb{R}I \mid \forall \theta \in wC_0, \langle \theta, x \rangle \leq \langle \theta, \underline{\dim} T^w \rangle\} = \underline{\dim} T^w - w \mathbb{R}_{\geq 0} I.$$

When  $\theta$  runs over the Tits cone, it generically belongs to a chamber, and we have just seen that in this case, the face  $P_\theta$  is a vertex. When  $\theta$  lies on a wall,  $P_\theta$  is a (possibly degenerate) edge. More precisely, if  $\theta$  lies on the facet that separates the chambers  $wC_0$  and  $(ws_i)C_0$ , with say  $\ell(ws_i) > \ell(w)$ , then  $(T_\theta^{\min}, T_\theta^{\max}) = (T^{ws_i}, T^w)$ . Results in [1] and [24] moreover assert that  $T^w/T^{ws_i}$  is the direct sum of copies of the module  $I_w \otimes_\Lambda S_i$ .

There is a similar description when  $\theta$  is in  $-C_T$ .

## 1.5 Vertical edges and partitions (in affine type)

From now on in this introduction, we focus on the case where  $\mathfrak{g}$  is of symmetric affine type, which in particular implies  $\mathfrak{g}$  is of untwisted affine type.

The root system for  $\mathfrak{g}$  decomposes into real and imaginary roots  $\Phi = \Phi^{\text{re}} \sqcup \mathbb{Z}_{\neq 0} \delta$ ; the real roots are the conjugate of the simple roots under the Weyl group action, whereas the imaginary roots are fixed under this action. The Tits cone is  $C_T = \{\theta : \mathbb{R}I \rightarrow \mathbb{R} \mid \langle \theta, \delta \rangle > 0\} \cup \{0\}$ .

We set  $\mathfrak{t}^* = \mathbb{R}I/\mathbb{R}\delta$ . The projection  $\pi : \mathbb{R}I \rightarrow \mathfrak{t}^*$  maps  $\Phi^{\text{re}}$  onto the “spherical” root system  $\Phi^s$ , whose Dynkin diagram is obtained from that of  $\mathfrak{g}$  by removing a zero node (a.k.a. extending vertex). The rank of  $\Phi^s$  is  $r = \dim \mathfrak{t}^*$ , which is also the multiplicity of the imaginary roots.

The vector space  $\mathfrak{t}$  identifies with the hyperplane  $\{\theta : \mathbb{R}I \rightarrow \mathbb{R} \mid \theta(\delta) = 0\}$  of the dual of  $\mathbb{R}I$ . The root system  $\Phi^s \subseteq \mathfrak{t}^*$  defines an hyperplane arrangement in  $\mathfrak{t}$ , called the spherical Weyl fan. The open cones in this fan will be called the spherical Weyl chambers. Together, this fan and the hyperplane arrangement that the real roots define in  $C_T \cup (-C_T)$  make up a (non locally-finite) fan in the dual of  $\mathbb{R}I$ , which we call the affine Weyl fan and which we denote by  $\mathscr{W}$ .

A set of simple roots in  $\Phi^s$  is a basis of  $\mathfrak{t}^*$ ; the elements of the corresponding dual basis of  $\mathfrak{t}$  are called the fundamental coweights. We denote by  $\Gamma$  the set of all fundamental coweights, for all possible choices of simple roots. Elements of  $\Gamma$  are called chamber coweights; the ray spanned by a chamber coweight is a minimal face of the spherical Weyl fan.

Now take a  $\Lambda$ -module  $T$ . The normal cone to  $\text{Pol}(T)$  at the vertex  $\underline{\dim} T^w$  (respectively,  $\underline{\dim} T_w$ ) contains  $wC_0$  (respectively,  $-w^{-1}C_0$ ). Altogether, these cones form a dense subset of the dual of  $\mathbb{R}I$ : this leaves no room for other vertices. This analysis also shows that the normal fan to  $\text{Pol}(T)$  is a coarsening of  $\mathscr{W}$ .

We then see that the edges of  $\text{Pol}(T)$  point in directions orthogonal to one-codimensional faces of  $\mathcal{W}$ , that is, parallel to roots. Since we have described in the previous section the edges that point in real root directions, we just have to understand the edges that are parallel to  $\delta$ . More generally, we are interested in describing the faces parallel to  $\delta$ .

Pick  $\theta \in \mathfrak{t}$  and look at the face  $P_\theta = \{x \in \text{Pol}(T) \mid \langle \theta, x \rangle = \psi_{\text{Pol}(T)}(\theta)\}$ . As we saw in Section 1.3, this face is the HN polytope of  $T_\theta^{\max}/T_\theta^{\min}$ , computed relative to the category  $\mathcal{R}_\theta$ . It turns out that given a face  $F$  of the spherical Weyl fan, the category  $\mathcal{R}_\theta$  is the same for all  $\theta \in F$ ; we record this independence in the notation by writing  $\mathcal{R}_F$  for  $\mathcal{R}_\theta$ .

We need one more definition: for  $\gamma \in \Gamma$ , we say that a  $\Lambda$ -module is a  $\gamma$ -core if it belongs to  $\mathcal{R}_\theta$  for all  $\theta \in \mathfrak{t}$  sufficiently close to  $\gamma$ . In other words, the category of  $\gamma$ -cores is the intersection of the categories  $\mathcal{R}_C$ , taken over all Weyl chambers  $C$  such that  $\gamma \in \overline{C}$ .

For each  $\nu \in \mathbb{N}I$ , the set of indecomposable modules is a constructible subset of  $\Lambda(\nu)$ . It thus makes sense to ask if the general point of an irreducible subset of  $\Lambda(\nu)$  is indecomposable. Similarly, the set of modules that belong to  $\mathcal{R}_C$  is an open subset of  $\Lambda(n\delta)$ , so we may ask if the general point of an irreducible subset of  $\Lambda(n\delta)$  is in  $\mathcal{R}_C$ . We will show the following theorems.

**Theorem 1.1** *For each integer  $n \geq 1$  and each  $\gamma \in \Gamma$ , there is a unique irreducible component  $Z$  of  $\Lambda(n\delta)$  such that the general point  $T$  in  $Z$  is an indecomposable  $\gamma$ -core.*

We denote by  $I(\gamma, n)$  this component.

**Theorem 1.2** *Let  $n$  be a positive integer and let  $C$  be a spherical Weyl chamber. There are exactly  $r$  irreducible components of  $\Lambda(n\delta)$  whose general point is an indecomposable module in  $\mathcal{R}_C$ . These components are the  $I(\gamma, n)$ , for  $\gamma \in \Gamma \cap \overline{C}$ .*

In Theorem 1.2, the multiplicity  $r$  of the root  $n\delta$  materializes as a number of irreducible components.

Now let  $b \in B(-\infty)$  and pick  $\theta$  in a spherical Weyl chamber  $C$ . Let  $T$  be a general point of  $\Lambda_b$  and let  $X = T_\theta^{\max}/T_\theta^{\min}$ . By Crawley-Boevey and Schröer's version of the Krull-Schmidt Theorem [17], we see that there exist positive integers  $n_k$  and irreducible components  $Z_k \subseteq \Lambda(n_k\delta)$  such that  $X = X_1 \oplus \cdots \oplus X_\ell$ , with  $X_k$  indecomposable and a general point in  $Z_k$ . By Theorem 1.2, we see that  $Z_k = I(\gamma_k, n_k)$  with  $\gamma_k \in \Gamma \cap \overline{C}$ . Gathering the  $n_k$  for each  $\gamma$ , we get a tuple of partitions  $(\lambda_\gamma)_{\gamma \in \Gamma \cap \overline{C}}$ . Moreover, we will show that the partition  $\lambda_\gamma$  depends only on  $b$  and  $\gamma$ , and not on the Weyl chamber  $C$ .

We are now ready to give the definition of the MV polytope of  $b$ : it is the datum  $\widetilde{\text{Pol}}(b)$  of the HN polytope  $\text{Pol}(T)$ , for  $T$  general in  $\Lambda_b$ , together with the above family of partitions  $(\lambda_\gamma)_{\gamma \in \Gamma}$ .

## 1.6 2-faces of MV polytopes

Let us now consider the 2-faces of our polytopes  $\text{Pol}(T)$ . Such a face is certainly of the form

$$P_\theta = \{x \in \text{Pol}(T) \mid \langle \theta, x \rangle = \psi_{\text{Pol}(T)}(\theta)\},$$

where  $\theta$  belongs to a two-codimensional face of  $\mathcal{W}$ . There are three possibilities, whether  $\theta$  belongs to  $C_T$ ,  $-C_T$  or  $\mathfrak{t}$ .

Suppose first that  $\theta \in C_T$ . Let  $w \in W$  be of minimal length such that  $\theta \in w\overline{C_0}$ . The root system  $\Phi_\theta = \Phi \cap (\ker \theta)$  is finite of rank 2, of type  $A_1 \times A_1$  or type  $A_2$ . The element  $w^{-1}$  turns  $\Phi_\theta$  to the standard Levi root system  $\Phi_J$ , where  $J = \{i \in I \mid \langle w^{-1}\theta, \alpha_i \rangle = 0\}$ . The full subgraph of  $(I, E)$  defined by  $J$  gives rise to a preprojective algebra  $\Lambda_J$ . The obvious surjective morphism  $\Lambda \rightarrow \Lambda_J$  induces an inclusion  $\Lambda_J\text{-mod} \hookrightarrow \Lambda\text{-mod}$ , whose image is the category  $\mathcal{R}_{w^{-1}\theta}$ .

With these notations, it can be shown that the tilting ideals  $I_w$  provide an equivalence of categories

$$\mathcal{R}_{w^{-1}\theta} \xrightleftharpoons[\text{Hom}_\Lambda(I_w, ?)]{I_w \otimes_\Lambda ?} \mathcal{R}_\theta,$$

whose action on the dimension-vectors is given by  $w$ . Therefore  $P_\theta$  is the image under  $w$  of the HN polytope of the  $\Lambda_J$ -module  $X = \text{Hom}_\Lambda(I_w, T_\theta^{\max}/T_\theta^{\min})$ .

Moreover, we will show that genericity is preserved in this construction as follows.

**Theorem 1.3** *If  $T$  is the general point in an irreducible component  $\Lambda_b$ , then  $X$  is a general point in an irreducible component of a nilpotent variety for  $\Lambda_J$ . If  $\Phi_J$  is of type  $A_2$ , this implies that, for general  $T$ ,  $P_\theta$  must obey the tropical Plücker relations from [32].*

A similar analysis can be done in the case where  $\theta$  is in  $-C_T$ .

Let us tackle the last case, where  $\theta \in \mathfrak{t}$ , that is, when  $\theta$  belongs to a face  $F$  of codimension one in the spherical Weyl fan. Then  $\Phi_\theta = \Phi \cap (\ker \theta)$  is an affine root system of type  $\tilde{A}_1$ . The face  $F$  separates two spherical Weyl chambers of  $\mathfrak{t}$ , say  $C'$  and  $C''$ . There are chamber coweights  $\gamma'$  and  $\gamma''$  such that  $\Gamma \cap \overline{C'} = (\Gamma \cap \overline{F}) \sqcup \{\gamma'\}$  and  $\Gamma \cap \overline{C''} = (\Gamma \cap \overline{F}) \sqcup \{\gamma''\}$ . Choose  $\theta' \in C'$  and  $\theta'' \in C''$ .

Assume that  $T$  is the general point of an irreducible component  $\Lambda_b$ . Then the modules  $T_{\theta'}^{\max}/T_{\theta'}^{\min}$  and  $T_{\theta''}^{\max}/T_{\theta''}^{\min}$  are described by tuples of partitions  $(\lambda_\gamma)_{\gamma \in \Gamma \cap \overline{C'}}$  and  $(\lambda_\gamma)_{\gamma \in \Gamma \cap \overline{C''}}$ , respectively. Both these modules are subquotients of  $T_\theta^{\max}/T_\theta^{\min}$ , so this latter contains the information about the partitions  $\lambda_\gamma$  for all  $\gamma \in (\Gamma \cap \overline{F}) \sqcup \{\gamma', \gamma''\}$ .

**Theorem 1.4** *Let  $Q$  be the polytope obtained by shortening all the vertical edges of  $P_\theta$  by  $\sum_{\gamma \in \Gamma \cap \overline{F}} |\lambda_\gamma| \delta$ . Then  $Q$ , equipped with the two partitions  $\lambda_{\gamma'}$  and  $\lambda_{\gamma''}$ , is an MV polytope of type  $\tilde{A}_1$ .*

Each of the remaining partitions  $(\lambda_\gamma)_{\gamma \in \Gamma \cap \overline{F}}$  can be thought of as an MV polytope of type  $\tilde{A}_0$ . Putting this together, we can thus say that the face  $P_\theta$ , together with all the partitions  $(\lambda_\gamma)_{\gamma \in (\Gamma \cap \overline{F}) \sqcup \{\gamma', \gamma''\}}$ , is (the projection of) an MV polytope of type  $\tilde{A}_1 \times \tilde{A}_0^{r-1}$ .

The rough idea for the proof of Theorem 1.4 is to construct an embedding of  $\Pi\text{-mod}$  into  $\mathcal{R}_\theta$ , where  $\Pi$  is the completed preprojective algebra of type  $\tilde{A}_1$ ; this embedding depends on  $F$  and its essential image is large enough to capture a dense open subset in the relevant irreducible component of Lusztig's variety. In this construction, we were inspired by the work of Kimura [35] who produced analogous embeddings in the quiver setting.

So the final picture is the following. Let  $\mathcal{MV}$  be the set of all lattice convex polytopes  $P$  in  $\mathbb{R}I$ , equipped with a family of partitions  $(\lambda_\gamma)_{\gamma \in \Gamma}$ , such that:

- The normal fan to  $P$  is a coarsening of the Weyl fan  $\mathcal{W}$ .
- To each spherical Weyl chamber  $C$  corresponds a vertical edge of  $P$ ; the difference between the two endpoints of this edge is equal to  $\sum_{\gamma \in \Gamma \cap \overline{C}} |\lambda_\gamma| \delta$ .
- A non-vertical 2-face of  $P$  is an MV polytope of type  $A_1 \times A_1$  or  $A_2$ ; in the latter case, this means that its shape obeys the tropical Plücker relations.
- Each vertical 2-face of  $P$  is an MV polytope of type  $\tilde{A}_1 \times \tilde{A}_0^{r-1}$ .

At the end of Section 1.5, we associated an element  $\widetilde{\text{Pol}}(b)$  of  $\mathcal{MV}$  to each  $b \in B(-\infty)$ .

**Theorem 1.5** *The map  $\widetilde{\text{Pol}} : B(-\infty) \rightarrow \mathcal{MV}$  is bijective.*

This theorem provides a complete characterization of the polytopes  $\widetilde{\text{Pol}}(b)$ . It also says that the datum of  $\widetilde{\text{Pol}}(b)$  determines  $b$ .

In a companion paper [4], we will provide a combinatorial description of MV polytopes of type  $\tilde{A}_1$ . With that result in hand, the above conditions give an explicit characterization of affine MV polytopes.



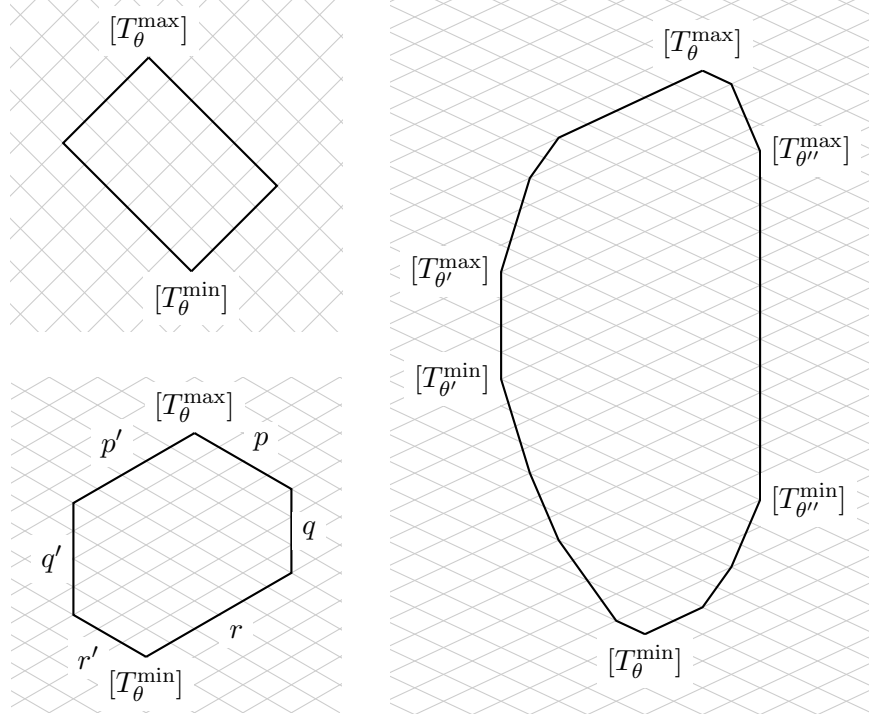


Figure 1: Examples of 2-faces of affine MV polytopes. These faces are of type  $A_1 \times A_1$  (top left),  $A_2$  (bottom left), and  $\tilde{A}_1$  (right). In type  $A_2$ , the tropical Plücker relation is  $q' = \min(p, r)$ . Note that the edges are parallel to root directions. In type  $\tilde{A}_1$ , the vertical edges are parallel to the imaginary roots and correspond to the modules  $T_{\theta'}^{\max}/T_{\theta'}^{\min}$  and  $T_{\theta''}^{\max}/T_{\theta''}^{\min}$  (notation as in the text above); for the picture drawn, the decorations are  $\lambda_{\gamma'} = (3, 1)$  and  $\lambda_{\gamma''} = (4, 2, 1, 1, 1, 1, 1, 1, 1)$ .

## 1.7 Lusztig data

As explained at the end of Section 1.1, for a finite dimensional  $\mathfrak{g}$ , the MV polytope  $\text{Pol}(b)$  of an element  $b \in B(-\infty)$  geometrically encodes all the Lusztig data of  $b$ .

In more detail, let  $N$  be the number of positive roots. Each reduced decomposition of the longest element  $w_0$  of  $W$  provides a PBW basis of the quantum group  $U_q(\mathfrak{n}_+)$ , which goes to the basis  $B(-\infty)$  at the limit  $q \rightarrow 0$ . To an element  $b \in B(-\infty)$ , one can therefore associate many PBW monomials, one for each PBW basis. In other words, one can associate to  $b$  many elements of  $\mathbb{N}^N$ , one for each reduced decomposition of  $w_0$ . These elements in  $\mathbb{N}^N$  are called the Lusztig data of  $b$ . A reduced decomposition of  $w_0$  specifies a path in the 1-skeleton of

$\text{Pol}(b)$  that connects the top vertex to the bottom one, and the corresponding Lusztig datum materializes as the lengths of the edges of this path.

With this in mind, we now explain that when  $\mathfrak{g}$  is of affine type, our MV polytopes  $\widetilde{\text{Pol}}(b)$  provide a fair notion of Lusztig data.

To this aim, we first note that a reasonable analog of the reduced decompositions of  $w_0$  is certainly the notion of “total reflection order” (Dyer) or “convex order” (Ito), see [14, 25]. By definition, this is a total order  $\preceq$  on  $\Phi_+$  such that

$$(\alpha + \beta \in \Phi_+ \text{ and } \alpha \preceq \beta) \implies \alpha \preceq \alpha + \beta \preceq \beta.$$

(Unfortunately, the convexity relation implies that  $m\delta \preceq n\delta$  for any positive integers  $m$  and  $n$ . We therefore have to accept that  $\preceq$  is only a preorder; this blemish is however limited to the imaginary roots.)

A convex order  $\preceq$  splits the positive real roots in two parts: those that are greater than  $\delta$  and those that are smaller. One easily shows that the projection  $\pi : \mathbb{R}I \rightarrow \mathfrak{t}^*$  maps  $\{\beta \in \Phi_+ \mid \beta \succ \delta\}$  onto a positive system of  $\Phi^s$ . In other words, there exists  $\theta \in \mathfrak{t}$  such that

$$\forall \beta \in \Phi_+^{\text{re}}, \quad \beta \succ \delta \iff \langle \theta, \beta \rangle > 0. \quad (1.1)$$

Given such a convex order  $\preceq$ , we will construct a functorial filtration  $(T_{\succcurlyeq \alpha})_{\alpha \in \Phi_+}$  on each finite dimensional  $\Lambda$ -module  $T$ , such that each  $\underline{\dim} T_{\succcurlyeq \alpha}$  is a vertex of  $\text{Pol}(T)$ . The family of dimension-vectors  $(\underline{\dim} T_{\succcurlyeq \alpha})_{\alpha \in \Phi_+}$  are the vertices along a path in the 1-skeleton of  $\text{Pol}(T)$  connecting the top vertex and bottom vertices. The lengths of the edges in this path form a family of natural numbers  $(n_\alpha)_{\alpha \in \Phi_+}$ , defined by the relation  $\underline{\dim} T_{\succcurlyeq \alpha} / T_{\succcurlyeq \alpha} = n_\alpha \alpha$ .

Choosing  $\theta \in \mathfrak{t}$  satisfying (1.1), we have  $T_{\succcurlyeq \delta} / T_{\succcurlyeq \delta} = T_\theta^{\max} / T_\theta^{\min}$ . Fix  $b \in B(-\infty)$  and recall the analysis carried after Theorem 1.2. For  $T$  general in  $\Lambda_b$ , the module  $T_\theta^{\max} / T_\theta^{\min}$  provides a tuple of partitions  $(\lambda_\gamma)_{\gamma \in \Gamma \cap \overline{C}}$ , where  $C$  is the spherical Weyl chamber containing  $\theta$ .

Thus, to each  $b \in B(-\infty)$ , we can associate the pair  $\Omega_{\preceq}(b)$  consisting in the family of integers  $(n_\alpha)_{\alpha \in \Phi_+^{\text{re}}}$ , and the family of partitions  $(\lambda_\gamma)_{\gamma \in \Gamma \cap \overline{C}}$ . All this information can be read from  $\widetilde{\text{Pol}}(b)$ . We call  $\Omega_{\preceq}(b)$  the Lusztig datum of  $b$  in direction  $\preceq$ . Denoting the set of all partitions by  $\mathcal{P}$ , we then have:

**Theorem 1.6** *The map  $\Omega_{\preceq} : B(-\infty) \rightarrow \mathbb{N}^{(\Phi_+^{\text{re}})} \times \mathcal{P}^{\Gamma \cap \overline{C}}$  is bijective.*

Let us conclude by a few remarks.

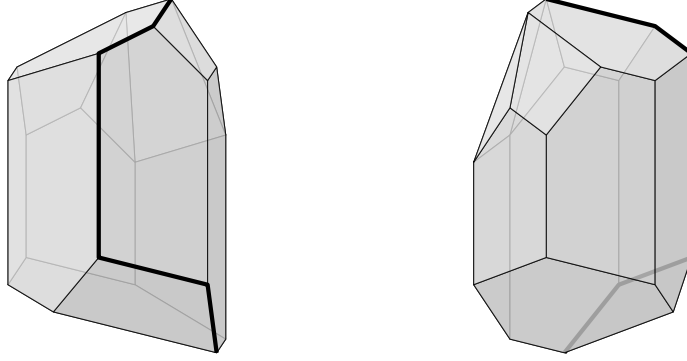


Figure 2: Two views of the same affine MV polytope  $\tilde{P}(b)$  of type  $\tilde{A}_2$ . The thick line goes successively through the points  $\nu_0 = 0$ ,  $\nu_1 = \nu_0 + (\alpha_0 + 2\alpha_1 + 2\alpha_2)$ ,  $\nu_2 = \nu_1 + 2\alpha_1$ ,  $\nu_3 = \nu_2 + 5\delta$ ,  $\nu_4 = \nu_3 + (\alpha_0 + \alpha_2)$ ,  $\nu_5 = \nu_4 + 2\alpha_0$ . The length of the edges of this line, together with the two partitions  $\lambda_{\varpi_1} = (1, 1)$  and  $\lambda_{\varpi_2} = (2, 1)$ , form the Lusztig datum  $\Omega_{\preccurlyeq}(b)$  relative to any convex order  $\preccurlyeq$  such that  $\alpha_0 \prec \alpha_0 + \alpha_2 \prec \mathbb{Z}_{>0}\delta \prec \alpha_1 \prec \alpha_0 + 2\alpha_1 + 2\alpha_2$ . The other vertices were calculated using the conditions on the 2-faces. The MV polytope  $\tilde{P}(b)$  includes the data of  $\lambda_\gamma$  for all chamber weights  $\gamma$ , and in this example, the rest of this decoration is given by  $\lambda_{s_1\omega_1} = (1, 1)$ ,  $\lambda_{s_2s_1\omega_1} = (0)$ ,  $\lambda_{s_2\omega_2} = (2, 1)$  and  $\lambda_{s_1s_2\omega_2} = (2, 1)$ .

- (i) The MV polytope  $\widetilde{\text{Pol}}(b)$  contains the information of all Lusztig data of  $b$ , for all convex orders. This is in complete analogy with the situation in the case where  $\mathfrak{g}$  is finite dimensional. The conditions on the 2-faces imposed by Theorem 1.5 say how the Lusztig datum varies when the convex order changes; they can be regarded as the analog in the affine type case of Lusztig piecewise linear bijections.
- (ii) The knowledge of a single Lusztig datum of  $b$ , for just one convex order, allows one to reconstruct the irreducible component  $\Lambda_b$ . This fact is indeed an ingredient of the proof of injectivity in Theorem 1.5.
- (iii) Through the map  $\widetilde{\text{Pol}}$ , the set  $\mathcal{MV}$  of MV polytopes acquires the structure of a crystal, isomorphic to  $B(-\infty)$ . This structure can be read from the Lusztig data. Specifically, if  $\alpha_i$  is the smallest element of the order  $\preccurlyeq$ , then  $\varphi_i(b)$  is the  $\alpha_i$ -coordinate of  $\Omega_{\preccurlyeq}(b)$ , and the operators  $\tilde{e}_i$  and  $\tilde{f}_i$  act by incrementing or decrementing this coordinate.
- (iv) As mentioned at the beginning of Section 1.2, Beck in [6] and Ito in [27] construct PBW bases of  $U_q(\mathfrak{n}_+)$  for  $\mathfrak{g}$  of affine type. An element in one of these bases is a monomial in root vectors, the product being computed according to a convex order  $\preccurlyeq$ . To describe a monomial, one needs an integer for each real root  $\alpha$  and a  $r$ -tuple of integers for each imaginary root  $n\delta$ , so in total, monomials in a PBW basis are indexed by  $\mathbb{N}^{(\Phi_+^{\text{re}})} \times \mathcal{P}^r$ .

Moreover, such a PBW basis goes to  $B(-\infty)$  at the limit  $q \rightarrow 0$ . (This fact has been established in [7] for Beck's bases, and the result can probably be extended to Ito's more general bases by using [42] or [50].) In the end, we get a bijection between  $B(-\infty)$  and  $\mathbb{N}^{(\Phi_+^{\text{re}})} \times \mathcal{P}^r$ . We expect that this bijection is our map  $\Omega_{\leq}$ .

## 1.8 Plan of the paper

Section 2 recalls combinatorial notions and facts related to root systems. We emphasize the notion of biconvex subsets, which is crucial to the study of convex orders and to the definition of the functorial filtration  $(T_{\succ \alpha})_{\alpha \in \Phi_+}$  mentioned in Section 1.7.

Section 3 is devoted to generalities about HN polytopes in abelian categories.

In Section 4, we recall known facts about preprojective algebras and Lusztig's nilpotent varieties. We also prove that cutting a  $\Lambda$ -module according to a torsion pair is an operation that preserves genericity.

In Section 5, we exploit the tilting theory on  $\Lambda\text{-mod}$  to define and study the submodules  $T^w$  mentioned in Section 1.4. An important difference with the works of Iyama, Reiten and al. and of Geiß, Leclerc and Schröer is the fact that we are not primarily interested in the small slices that form the categories  $\text{Sub}(\Lambda/I_w)$  (notation of Iyama, Reiten and al.) or  $\mathcal{C}_w$  (notation of Geiß, Leclerc and Schröer), but rather at also controlling the remainder. Moreover, we track the tilting theory at the level of the irreducible components of Lusztig's nilpotent varieties and interpret the result in term of crystal operations.

In Section 6, we construct embeddings of  $\Pi\text{-mod}$  into  $\Lambda\text{-mod}$ , where  $\Pi$  is the completed preprojective algebra of type  $\tilde{A}_1$ . The data needed to define such an embedding is a pair  $(S, R)$  of rigid orthogonal bricks in  $\Lambda\text{-mod}$  satisfying  $\dim \text{Ext}_{\Lambda}^1(S, R) = \dim \text{Ext}_{\Lambda}^1(R, S) = 2$ . The key ingredient in the construction is the 2-Calabi-Yau property of  $\Lambda\text{-mod}$ .

The final Section 7 deals with the specificities of the affine type case. All the results concerning the vertical edges and faces and all the results about the cores  $I(\gamma, n)$  are stated and proved there.

## 1.9 Thanks

We warmly thank Claire Amiot for suggesting that the reflection functors used in [5] could be related to those defined in [1]. Thanks to her help, the present writing of Section 5 takes into account the current literature [13, 24, 52]. We also thank Thomas Dunlap for sharing with us his ideas about affine MV polytopes and for providing us with his PhD thesis [18]. Last

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## 1.10 Summary of the main notations

$\mathbb{N} = \{0, 1, 2, \dots\}$ .

$\mathcal{P}$  the set of partitions.

$\text{Irr } \mathcal{A}$  the set of isomorphism classes of simple objects in  $\mathcal{A}$ , an essentially small abelian category.

$\text{Pol}(T) \subseteq K_0(\mathcal{A})_{\mathbb{R}}$  the HN polytope of an object  $T \in \mathcal{A}$ .

$P_{\theta} = \{x \in P \mid \langle \theta, x \rangle = m\}$  the face of a HN polytope  $P$ , where  $\theta \in \text{Hom}(K_0(\mathcal{A}), \mathbb{R})$  and  $m$  is the maximal value of  $\theta$  on  $P$ .

$K$  the base field for representation of quivers and preprojective algebras.

$(I, E)$  a finite graph, without loops (encoding a symmetric generalized Cartan matrix).

$\mathfrak{g}$  the corresponding symmetric Kac-Moody algebra.

$\mathfrak{n}_+$  the upper nilpotent subalgebra of  $\mathfrak{g}$ .

$Q = (I, \Omega)$  an orientation of  $(I, E)$  (thus  $Q$  is a quiver).

$H = \Omega \sqcup \Omega^*$  the set of edges of the double quiver  $\overline{Q}$ .

$s, t : H \rightarrow I$  the source and target maps.

$\Lambda$  the completed preprojective algebra of  $Q$ .

$\Lambda\text{-mod}$  the category of finite dimensional  $\Lambda$ -modules.

$\Phi$  the root system of  $\mathfrak{g}$ .

$\{\alpha_i \mid i \in I\}$  the standard basis of  $\Phi$ .

$\Phi = \Phi_+ \sqcup \Phi_-$  the positive and negative roots with respect to this basis.

$W = \langle s_i \mid i \in I \rangle$  the Weyl group.

$\ell : W \rightarrow \mathbb{N}$  the length function.

$\mathbb{Z}I = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$  the root lattice,  $\mathbb{R}I = \mathbb{Z}I \otimes_{\mathbb{Z}} \mathbb{R}$ , and its dual  $(\mathbb{R}I)^*$ .

$(, ) : \mathbb{Z}I \times \mathbb{Z}I \rightarrow \mathbb{Z}$  the  $W$ -invariant symmetric bilinear form;

(normalized so that  $(\alpha, \alpha) = 2$  for each real root  $\alpha$ ).

$\omega_i : \mathbb{R}I \rightarrow \mathbb{R}$  the  $i$ -th coordinate on  $\mathbb{R}I$ ; thus  $(\omega_i)_{i \in I}$  is the dual basis of  $(\alpha_i)_{i \in I}$ .

$C_0 = \{\theta \in (\mathbb{R}I)^* \mid \langle \theta, \alpha_i \rangle > 0\}$  the dominant Weyl chamber.

$C_T = \bigcup_{w \in W} \overline{C_0}$  the Tits cone.

$F_J = \{\theta \in (\mathbb{R}I)^* \mid \forall j \in J, \langle \theta, \alpha_j \rangle = 0 \text{ and } \forall i \in I \setminus J, \langle \theta, \alpha_i \rangle > 0\}$ , for  $J \subseteq I$ .

$\Phi_J$  and  $W_J$ , the root subsystem and the parabolic subgroup defined by  $J \subseteq I$ .

$w_J$  the longest element in  $W_J$ , when the latter is finite.

$\text{ht} : \mathbb{Z}I \rightarrow \mathbb{Z}$  the linear form such that  $\text{ht}(\alpha_i) = 1$  for each  $i \in I$ .

$N_w = \Phi_+ \cap w\Phi_-$ , for  $w \in W$ ;  
 (thus  $N_w = \{s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \mid 1 \leq k \leq \ell\}$  for any reduced expression  $w = s_{i_1} \cdots s_{i_\ell}$ ).  
 $\Pi$  the preprojective algebra of type  $\widetilde{A}_1$ .

In the case of an affine root system:

$\delta$  the positive indecomposable imaginary root.

$\mathfrak{t}^* = \mathbb{R}I/\mathbb{R}\delta$ .

$\pi : \mathbb{R}I \rightarrow \mathfrak{t}^*$  the projection modulo  $\mathbb{R}\delta$ .

$\Phi^s = \pi(\Phi^{\text{re}})$  the spherical (finite) root system.

$\iota : \Phi^s \rightarrow \Phi_+^{\text{re}}$  the ‘minimal’ lift, a right inverse of  $\pi$ .

$r = \dim \mathfrak{t}^*$  the rank of  $\Phi^s$ .

$\mathfrak{t} = \{\theta \in (\mathbb{R}I)^* \mid \langle \theta, \delta \rangle = 0\}$  the dual of  $\mathfrak{t}^*$ .

$\Gamma \subseteq \mathfrak{t}$  the set of all spherical chamber coweights.

$\mathscr{W}$  the Weyl fan on  $(\mathbb{R}I)^*$ , completed on  $\mathfrak{t}$  by the spherical Weyl fan.

$Q^\vee \subseteq \mathfrak{t}$  the coroot lattice, spanned over  $\mathbb{Z}$  by the elements  $(\alpha_i, ?)$ .

$t_\lambda \in W$  the translation, for  $\lambda \in Q^\vee$ ; thus  $t_\lambda(\nu) = \nu - \langle \lambda, \nu \rangle \delta$  for each  $\nu \in \mathbb{R}I$ .

$W_0 = W/Q^\vee$  the image of  $W$  in  $\text{GL}(\mathfrak{t}^*)$ .

$\mathscr{V} = \{A \subseteq \Phi_+ \mid A \text{ biconvex}\}$ .

And after having chosen a zero node 0 (extending vertex) in the extended Dynkin diagram:

$I_0 = I \setminus \{0\}$  the vertices of the (finite type) Dynkin diagram.

$\{\pi(\alpha_i) \mid i \in I_0\}$  a preferred system of simple roots for  $\Phi^s$ .

$(\varpi_i)_{i \in I_0}$  the (spherical) fundamental coweights, a basis of  $\mathfrak{t}$ .

$C_0^s = \sum_{i \in I_0} \mathbb{R}_{>0} \varpi_i$  the dominant spherical Weyl chamber.

$W_{I_0} = \langle s_i \mid i \in I_0 \rangle$ , a distinguished lift of  $W_0$  inside  $W$ .

Geometry:

The set of irreducible components of a topological space  $X$  is denoted by  $\text{Irr } X$ . If  $Z$  is an irreducible topological space, then we say that a propriety  $P(x)$  depending on a point  $x \in Z$  holds for  $x$  general in  $Z$  if the set of points of  $Z$  at which  $P$  holds true contains a dense open subset of  $Z$ . We sometimes extend this vocabulary by simply saying ‘let  $x$  be a general point in  $Z$ ’; in this case, it is understood that we plan to impose finitely many such conditions  $P$ .

## 2 Combinatorics of root systems and of MV polytopes

### 2.1 General setup

Let  $(I, E)$  be a finite graph, without loops: here  $I$  is the set of vertices and  $E$  is the set of edges. We denote by  $\mathbb{Z}I$  the corresponding root lattice and we denote its canonical basis by

$\{\alpha_i \mid i \in I\}$ . We endow it with the symmetric bilinear form  $(, ) : \mathbb{Z}I \times \mathbb{Z}I \rightarrow \mathbb{Z}$ , given by  $(\alpha_i, \alpha_i) = 2$  for any  $i$ , and for  $i \neq j$ ,  $(\alpha_i, \alpha_j)$  is the negative of the number of edges between the vertices  $i$  and  $j$  in the graph  $(I, E)$ . The Weyl group is the subgroup of  $\text{GL}(\mathbb{Z}I)$  generated by the simple reflections  $s_i : \alpha_j \mapsto \alpha_j - (\alpha_j, \alpha_i)\alpha_i$ ; this is in fact a Coxeter system, whose length function is denoted by  $\ell$ . Lastly, we denote by  $\mathbb{N}I$  the set of all linear combinations of the  $\alpha_i$  with coefficients in  $\mathbb{N}$  and we denote by  $\text{ht} : \mathbb{Z}I \rightarrow \mathbb{Z}$  the linear form that maps each  $\alpha_i$  to 1.

The matrix with entries  $(\alpha_i, \alpha_j)$  is a symmetric generalized Cartan matrix, hence gives rise to a Kac-Moody algebra  $\mathfrak{g}$  and a root system  $\Phi$ . The latter is a  $W$ -stable subset of  $\mathbb{Z}I$ , which can be split into positive and negative roots  $\Phi = \Phi_+ \sqcup \Phi_-$  and into real and imaginary roots  $\Phi = \Phi^{\text{re}} \sqcup \Phi^{\text{im}}$ .

Given a subset  $J \subseteq I$ , we can look at the root system  $\Phi_J = \Phi \cap \text{span}_{\mathbb{Z}}\{\alpha_j \mid j \in J\}$ . Its Weyl group is the parabolic subgroup  $W_J = \langle s_j \mid j \in J \rangle$  of  $W$ . If  $W_J$  is finite, then it has a longest element, which we denote by  $w_J$ . An element  $u \in W$  is called  $J$ -reduced on the right if  $\ell(us_j) > \ell(u)$  for each  $j \in J$ . If  $u$  is  $J$ -reduced on the right, then  $\ell(uv) = \ell(u) + \ell(v)$  for all  $v \in W_J$ . Each right coset of  $W_J$  in  $W$  contains a unique element that is  $J$ -reduced on the right.

The Weyl group acts on  $\mathbb{R}I$  and on its dual  $(\mathbb{R}I)^*$ . The dominant chamber  $C_0$  and the Tits cone  $C_T$  are the convex cones in  $(\mathbb{R}I)^*$  defined as

$$C_0 = \{\theta : \mathbb{R}I \rightarrow \mathbb{R} \mid \forall i \in I, \langle \theta, \alpha_i \rangle > 0\} \quad \text{and} \quad C_T = \bigcup_{w \in W} w \overline{C_0}.$$

The closure  $\overline{C_0}$  is the disjoint union of faces

$$F_J = \{\theta \in (\mathbb{R}I)^* \mid \forall j \in J, \langle \theta, \alpha_j \rangle = 0 \text{ and } \forall i \in I \setminus J, \langle \theta, \alpha_i \rangle > 0\},$$

for  $J \subseteq I$ . The stabilizer of any point in  $F_J$  is precisely the parabolic subgroup  $W_J$ . Thus

$$\overline{C_0} = \bigsqcup_{J \subseteq I} F_J \quad \text{and} \quad C_T = \bigsqcup_{J \subseteq I} \bigsqcup_{w \in W/W_J} w F_J.$$

The disjoint union on the right endows  $C_T$  with the structure of a (locally finite) fan, which we call the Tits fan.

To an element  $w \in W$ , we associate the subset  $N_w = \Phi_+ \cap w\Phi_-$ . If  $w = s_{i_1} \cdots s_{i_\ell}$  is a reduced decomposition, then

$$N_w = \{s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \mid 1 \leq k \leq \ell\}.$$

The following result is well-known (see for instance Remark  $\clubsuit$  in [14]).

**Lemma 2.1** *For  $(u, v) \in W^2$ , the following three properties are equivalent:*

$$\ell(u) + \ell(v) = \ell(uv), \quad N_u \subseteq N_{uv}, \quad N_{u^{-1}} \cap N_v = \emptyset.$$

We also need the following result.

**Lemma 2.2** *Let  $J \subseteq I$  and let  $w \in W$ . If  $w$  is  $J$ -reduced on the right, then  $N_{w^{-1}} \cap \Phi_J = \emptyset$ .*

*Proof.* Let  $w = s_{i_\ell} \cdots s_{i_1}$  be a reduced expression in  $W$ . Suppose that  $N_{w^{-1}}$  contains an element of  $\Phi_J$ , say  $\beta = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$ . Since  $\beta \in \Phi_J$ , there is a reduced expression  $v = s_{j_1} \cdots s_{j_m}$  in  $W_J$  such that  $\beta = s_{j_1} \cdots s_{j_{m-1}} \alpha_{j_m}$ . Then

$$s_{i_k} s_{i_{k-1}} \cdots s_{i_1} s_{j_1} \cdots s_{j_{m-1}} s_{j_m} = s_{i_{k-1}} \cdots s_{i_1} s_{j_1} \cdots s_{j_{m-1}},$$

and a fortiori  $\ell(wv) < \ell(w) + \ell(v)$ . Thus  $w$  is not  $J$ -reduced on the right. This proves the lemma by contraposition.  $\square$

## 2.2 Setup in the affine type

In this paper, we are mostly concerned with the case where  $\mathfrak{g}$  is a symmetric affine Kac-Moody algebra. In this case, there exists  $\delta \in \mathbb{Z}I$  such that  $\Phi_+^{\text{im}} = \mathbb{Z}_{>0} \delta$ . We set  $\mathfrak{t}^* = \mathbb{R}I / \mathbb{R}\delta$  and we denote the natural projection by  $\pi : \mathbb{R}I \rightarrow \mathfrak{t}^*$ . Then  $\Phi^s = \pi(\Phi^{\text{re}})$  is a finite type root system in  $\mathfrak{t}^*$ , called the spherical root system. The Weyl group leaves  $\delta$  invariant, hence acts on  $\mathfrak{t}^*$ . The kernel of this action consists of translations  $t_\lambda$ , for  $\lambda$  in the coroot lattice  $Q^\vee$  of  $\Phi^s$ . The translation  $t_\lambda$  acts on  $\mathbb{R}I$  by  $t_\lambda \nu = \nu - \langle \lambda, \nu \rangle \delta$ . We denote by  $W_0$  the quotient of  $W$  by this subgroup of translations; it can be viewed as a subgroup of  $\text{GL}(\mathfrak{t}^*)$ .

A basis of  $\Phi^s$  is in particular a basis of  $\mathfrak{t}^*$ . The elements of the dual basis are called the fundamental coweights. We define a spherical chamber coweight as an element of  $\mathfrak{t}$  that is conjugate under  $W_0$  to a fundamental coweight and denote by  $\Gamma$  the set of spherical chamber coweights.

The dual vector space  $\mathfrak{t}$  of  $\mathfrak{t}^*$  is identified with  $\{\theta \in (\mathbb{R}I)^* \mid \langle \theta, \delta \rangle = 0\}$ . The root system  $\Phi^s$  defines an hyperplane arrangement in  $\mathfrak{t}$ , called the spherical Weyl fan. The open cones in this fan will be called the spherical Weyl chambers. The minimal faces of this fan, that is, the rays, are spanned by the (spherical) chamber coweights.

The Tits cone is  $C_T = \{\theta \in (\mathbb{R}I)^* \mid \langle \theta, \delta \rangle > 0\} \cup \{0\}$ . Thus  $(\mathbb{R}I)^*$  is covered by  $C_T$ ,  $-C_T$  and  $\mathfrak{t}$ . Gathering the faces of the Tits fan, their opposite, and the faces of the spherical Weyl fan, we get a (non-locally finite) fan on  $(\mathbb{R}I)^*$ . We call it the affine Weyl fan and denote it by  $\mathscr{W}$ .



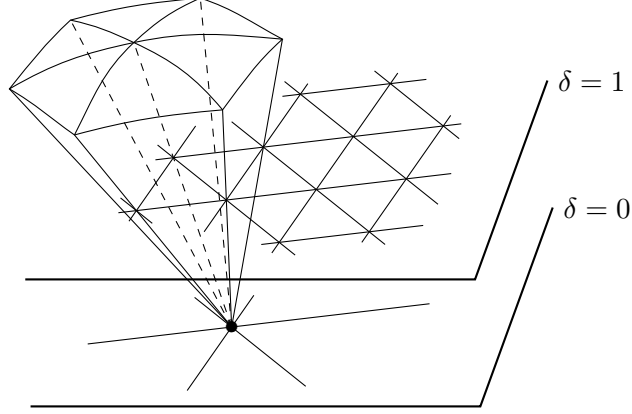


Figure 3: The upper half of the affine Weyl fan of type  $\tilde{A}_2$ . The intersection with the affine hyperplane of equation  $\delta = 1$  draws the familiar pattern of alcoves. On the hyperplane  $\delta = 0$ , one gets the spherical Weyl fan.

For each  $\alpha \in \Phi^s$ , we denote by  $\iota(\alpha) \in \Phi_+^{\text{re}}$  the unique positive real root such that  $\pi(\iota(\alpha)) = \alpha$  and  $\iota(\alpha) - \delta \notin \Phi_+^{\text{re}}$ . Thus  $\iota : \Phi^s \rightarrow \Phi_+^{\text{re}}$  is the “minimal” right inverse to  $\pi$ .

It is often convenient to embed the spherical root system  $\Phi^s$  in the affine root system  $\Phi$ . To do that, we choose an extending vertex  $0$  in  $I$  and we set  $I_0 = I \setminus \{0\}$ . Then the spherical Weyl group  $W_0$  can be identified with the parabolic subgroup  $W_{I_0} = \langle s_i \mid i \in I_0 \rangle$  of  $W$ . The spherical root system  $\Phi^s$  is now endowed with a set of simple roots, namely  $\{\pi(\alpha_i) \mid i \in I_0\}$ , whence a dominant spherical Weyl chamber  $C_0^s$ . The highest root  $\tilde{\alpha}$  of  $\Phi^s$  relative to this set of simple roots satisfies  $\delta = \alpha_0 + \iota(\tilde{\alpha})$ . We denote by  $\{\varpi_i \mid i \in I_0\}$  the basis of  $\mathfrak{t}$  dual to the basis  $\{\pi(\alpha_i) \mid i \in I_0\}$  of  $\mathfrak{t}^*$ .

### 2.3 Biconvex sets

A subset  $A \subseteq \Phi$  is said to be *clos* if the conditions  $\alpha \in A$ ,  $\beta \in A$ ,  $\alpha + \beta \in \Phi$  imply  $\alpha + \beta \in A$  (see [9], chapitre 6, §1, n° 7, Définition 4). A subset  $A \subseteq \Phi_+$  is said to be *biconvex* if both  $A$  and  $\Phi_+ \setminus A$  are *clos*. We will denote by  $\mathcal{V}$  the set of all biconvex subsets of  $\Phi_+$ .

*Remark 2.3.* Define a positive root system as a subset  $X \subseteq \Phi$  such that  $\Phi = X \sqcup (-X)$  and that the convex cone spanned in  $\mathbb{R}I$  by  $X$  is acute. Denote by  $\tilde{\mathcal{V}}$  the set of all positive root systems in  $\Phi$ . It can be shown that if  $\Phi$  is of finite or affine type, then the map  $X \mapsto X \cap \Phi_+$

is a bijection from  $\tilde{\mathcal{V}}$  onto  $\mathcal{V}$ . Worded another way, this means that, in finite or affine type, a set  $A \subseteq \Phi_+$  is biconvex if and only if the convex cone spanned by  $X = A \sqcup (-(\Phi_+ \setminus A))$  is acute, which is equivalent to saying that the convex cones spanned by  $A$  and by  $\Phi_+ \setminus A$  intersect only at the origin. This result gives a geometric flavor to the notion of biconvex subsets that is closer to the spirit of the present paper. However, in order to more easily refer to the literature, we use the above definition based on clos subsets.

*Examples 2.4.* (i) An increasing union or a decreasing intersection of biconvex subsets is itself biconvex.

(ii) Each finite biconvex subset of  $\Phi_+$  consists of real roots and is a  $N_w$ , with  $w \in W$  (see [14], Proposition 3.2). For convenience, we will say that a biconvex set  $A$  is cofinite if its complement  $\Phi_+ \setminus A$  is finite. Given  $w \in W$ , we set  $A_w = N_{w^{-1}}$  and  $A^w = \Phi_+ \setminus N_w$ . Thus the map  $w \mapsto A_w$  (respectively,  $w \mapsto A^w$ ) is a bijection from  $W$  onto the set of finite (respectively, cofinite) biconvex subsets of  $\Phi_+$ .

(iii) Each  $\theta \in (\mathbb{R}I)^*$  gives birth to two biconvex subsets

$$A_\theta^{\min} = \{\alpha \in \Phi_+ \mid \langle \theta, \alpha \rangle > 0\} \quad \text{and} \quad A_\theta^{\max} = \{\alpha \in \Phi_+ \mid \langle \theta, \alpha \rangle \geq 0\}.$$

(iv) The construction in (iii) can be refined to produce more biconvex subsets, by an inductive procedure on the rank of the root system. Specifically,  $\Phi \cap \ker \theta$  can be regarded as a root system in  $\ker \theta$  and each biconvex subset  $B$  in  $\Phi_+ \cap \ker \theta$  provides a biconvex subset  $A_\theta^{\min} \sqcup B$  in  $\Phi_+$ . Cellini and Papi's and Ito's works [14, 26] seem to indicate that this procedure yields all biconvex subsets when  $\Phi$  is of affine type.

**Proposition 2.5** *Let  $J \subseteq I$ , let  $\theta \in F_J$ , and let  $w \in W$ . Assume  $w$  is  $J$ -reduced on the right. Then  $A^w = A_{w\theta}^{\max}$  and  $A_w = A_{-w^{-1}\theta}^{\min}$ . In addition, if  $W_J$  is finite, then  $A^{ww_J} = A_{w\theta}^{\min}$  and  $A_{w_Jw} = A_{-w^{-1}\theta}^{\max}$ .*

*Proof.* Let  $J, \theta, w$  as in the statement of the theorem.

We have  $A^w = \{\alpha \in \Phi_+ \mid w^{-1}\alpha \in \Phi_+\}$  and  $A_{w\theta}^{\max} = \{\alpha \in \Phi_+ \mid \langle \theta, w^{-1}\alpha \rangle \geq 0\}$ . The inclusion  $A^w \subseteq A_{w\theta}^{\max}$  is straightforward. To show the reverse inclusion, we take  $\alpha \in \Phi_+ \setminus A^w$ , that is,  $\alpha \in N_w$ . Then  $\beta = -w^{-1}\alpha$  is in  $N_{w^{-1}}$ , in particular  $\beta \in \Phi_+$ , but  $\beta \notin \Phi_J$  by Lemma 2.2, and so  $\langle \theta, \beta \rangle > 0$ , which means that  $\alpha \notin A_{w\theta}^{\max}$ . We conclude that  $A^w = A_{w\theta}^{\max}$ .

Suppose now that  $W_J$  is finite. Then  $A^{ww_J} = \{\alpha \in \Phi_+ \mid w_Jw^{-1}\alpha \in \Phi_+\}$  and  $A_{w\theta}^{\min} = \{\alpha \in \Phi_+ \mid \langle \theta, w_Jw^{-1}\alpha \rangle > 0\}$ . The inclusion  $A_{w\theta}^{\min} \subseteq A^{ww_J}$  is straightforward. To show the reverse inclusion, we take  $\alpha \in A^{ww_J} \setminus A_{w\theta}^{\min}$ , if possible. Then  $w_Jw^{-1}\alpha$  necessarily belongs to  $\Phi_+ \cap \Phi_J$ ,

and so does  $\beta = -w^{-1}\alpha$ . Then  $\beta \in N_{w^{-1}} \cap \Phi_J$ , which contradicts Lemma 2.2. We conclude that  $A^{ww_J} = A_{w\theta}^{\min}$ .

The last two equalities  $A_w = A_{-w^{-1}\theta}^{\min}$  and  $A_{w_Jw} = A_{-w^{-1}\theta}^{\max}$  are proved in a similar fashion.  $\square$

For the rest of this section, we will assume that  $\Phi$  is of affine type.

**Lemma 2.6** *Let  $A \subseteq \Phi_+$  be a biconvex subset such that  $\delta \notin A$ .*

- (i)  *$A$  is the union of an increasing sequence of finite biconvex subsets.*
- (ii) *Let  $X = \pi(A)$ . Then  $X$  is contained in a positive root system of  $\Phi^s$  and  $\iota(X) \subseteq A$ .*

*Proof.* Assertion (i) is a direct application of Theorem 3.12 in [14]. Let us look at (ii). Since  $A$  is clos, so is  $X$ . Since  $\delta \notin A$ , we also have  $X \cap (-X) = \emptyset$ . By [9], chapitre 6, §1, n° 7, Proposition 22,  $X$  is contained in a positive root system of  $\Phi^s$ . Lastly, let  $\alpha \in X$ , and choose  $\beta \in A \cap \pi^{-1}(\alpha)$  of minimal height. Then  $\beta - \delta$  is not in  $A$ . It is not in  $\Phi_+ \setminus A$  either, for this latter is clos and contains  $\delta$  but not  $\beta$ . Therefore  $\beta - \delta \notin \Phi_+$ , which means that  $\beta = \iota(\alpha)$ . We have shown that  $\iota(X) \subseteq A$ .  $\square$

Taking complements in (i), we see that a biconvex subset that contains  $\delta$  is the intersection of a decreasing sequence of cofinite biconvex subsets.

An example of this approximation method is given by the following equalities, whose proof is left to the reader: for any  $\lambda \in Q^\vee$ , we have

$$A_\lambda^{\min} = \bigcup_{n \in \mathbb{N}} A_{t_{n\lambda}} \quad \text{and} \quad A_\lambda^{\max} = \bigcap_{n \in \mathbb{N}} A^{t_{n\lambda}}. \quad (2.1)$$

**Lemma 2.7** *Let  $\alpha \in \Phi_+^{\text{re}}$  and let  $A$  and  $B$  be two biconvex subsets such that  $B = A \sqcup \{\alpha\}$ . We assume that  $\delta \notin A$ . Then, for each finite subset  $X \subseteq A$ , there are finite biconvex subsets  $A' \subseteq A$  and  $B' \subseteq B$  such that  $X \subseteq A'$  and  $B' = A' \sqcup \{\alpha\}$ .*

*Proof.* Since  $A$  is the increasing union of finite biconvex subsets, one can find a biconvex  $A_0 \subseteq A$  that contains  $X$ . Similarly, one can find a finite biconvex  $B_0 \subseteq B$  that contains  $A_0 \cup \{\alpha\}$ . By Example 2.4 (ii), we can write  $A_0 = A_u$  and  $B_0 = A_{uv}$ , with  $(u, v) \in W^2$ . Lemma 2.1 says then that  $\ell(uv) = \ell(u) + \ell(v)$ . Let us write a reduced decomposition  $s_{i_1} \cdots s_{i_\ell}$  for  $v$ . There exists  $k$  such that  $\alpha = us_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$ . We then take  $A' = A_{us_{i_1} \cdots s_{i_{k-1}}}$  and  $B' = A_{us_{i_1} \cdots s_{i_k}}$ .  $\square$

**Lemma 2.8** (i) Let  $A$  and  $B$  be two biconvex subsets such that  $B = A \sqcup \mathbb{Z}_{>0}\delta$ . Then there is a positive system  $X \subseteq \Phi^s$  such that  $A = \{\alpha \in \Phi_+ \mid \pi(\alpha) \in X\}$ .

(ii) Let  $A \subseteq B$  be two biconvex subsets. Suppose that  $A$  is finite and that  $B$  is cofinite. Then there is a positive system  $X \subseteq \Phi^s$  such that  $A \subseteq \{\alpha \in \Phi_+ \mid \pi(\alpha) \in X\} \subseteq B$ .

*Proof.* Let us first show (i). We take  $A$  and  $B$  as in the statement to be proved. Let  $X = \pi(A)$  and  $Y = \pi(\Phi_+ \setminus B)$ . Certainly,  $\Phi^s = \pi(\Phi_+^{\text{re}}) = X \cup Y$ . In addition,  $X$  and  $Y$  are disjoint, because otherwise the inclusions  $\iota(X) \subseteq A$  and  $\iota(Y) \subseteq \Phi_+ \setminus B$  given by Lemma 2.6 (ii) would force  $A$  and  $\Phi_+ \setminus B$  to share a common element. Lastly,  $X$  and  $Y$  are clos. By [9], chapitre 6, §1, n° 7, Corollaire 1 to Proposition 20,  $X$  is a positive system in  $\Phi^s$ . Assertion (i) then follows from the observation that  $A \subseteq \pi^{-1}(X)$  and  $\pi(\Phi_+ \setminus A) \subseteq Y \cup \{0\}$ .

Now we consider (ii). By Example 2.4 (ii), there are  $(u, v) \in W^2$  such that  $A = A_u = N_{u^{-1}}$  and  $B = A^v = \Phi_+ \setminus N_v$ . The condition  $A \subseteq B$  means that  $N_{u^{-1}} \cap N_v = \emptyset$ , so  $\ell(uv) = \ell(u) + \ell(v)$  by Lemma 2.1. By Lemma 2.6 (ii),  $\pi(N_{uv})$  is contained in a positive root system  $\Phi^s$ , say  $Y$ . Since  $\pi : \mathbb{R}I \rightarrow \mathfrak{t}^*$  is  $W$ -equivariant,  $\pi(u^{-1}N_{uv})$  is contained in  $X = -u^{-1}Y$ . From the equality  $u^{-1}N_{uv} = u^{-1}(N_u \sqcup uN_v) = (-N_{u^{-1}}) \sqcup N_v$ , we deduce that  $\pi(N_{u^{-1}}) \subseteq X$  and that  $\pi(N_v) \cap X = \emptyset$ . Therefore  $N_{u^{-1}} \subseteq \{\alpha \in \Phi_+ \mid \pi(\alpha) \in X\} \subseteq \Phi_+ \setminus N_v$ .  $\square$

## 2.4 Convex orders

One motivation for studying biconvex subsets comes from the notion of “convex order” on  $\Phi_+$ . Specifically, a preorder  $\preccurlyeq$  on  $\Phi_+$  is called a convex order if for all  $(\alpha, \beta) \in \Phi_+^2$ , the three following conditions hold:

$$\begin{aligned} \alpha \preccurlyeq \beta \text{ or } \beta \preccurlyeq \alpha, \\ (\alpha + \beta \in \Phi_+ \text{ and } \alpha \preccurlyeq \beta) \implies \alpha \preccurlyeq \alpha + \beta \preccurlyeq \beta, \\ (\alpha \preccurlyeq \beta \text{ and } \beta \preccurlyeq \alpha) \iff \alpha \text{ and } \beta \text{ are proportional.} \end{aligned}$$

In this section, we restrict to affine type. In this case, in the last condition above,  $\alpha$  and  $\beta$  are proportional if and only if they are equal or they are both imaginary.

A terminal section for a convex order  $\preccurlyeq$  is a subset  $A \subseteq \Phi_+$  such that

$$(\alpha \in A \text{ and } \alpha \preccurlyeq \beta) \implies \beta \in A.$$

We denote the set of terminal sections of  $\preccurlyeq$  by  $\mathcal{U}(\preccurlyeq)$ . The following result, implicit in [14, 25], provides the link between biconvex subsets and convex orders. We leave its (routine) proof as an exercise for the reader.

**Lemma 2.9** *For each convex order  $\preceq$ , the set  $\mathcal{U}(\preceq)$  is a maximal totally ordered subset of  $\mathcal{V}$ . The map  $\mathcal{U}$  is injective.*

*Remarks 2.10.* (i) It is known that any biconvex subset is the terminal section of a convex order (Corollary 3.13 in [14]).

(ii) Let us say that a pair  $(A, B)$  of biconvex subsets is adjacent if  $A \subsetneq B$  and if there is not a biconvex subset  $C$  such that  $A \subsetneq C \subsetneq B$ . Each  $(w, i) \in W \times I$  with  $\ell(ws_i) > \ell(w)$  gives such a pair, namely  $(N_w, N_{ws_i})$ ; indeed, one here observes that  $N_{ws_i} = N_w \sqcup \{w\alpha_i\}$ , so there is no room between  $N_w$  and  $N_{ws_i}$ . Using Lemma 2.1, one easily shows that any adjacent pair of finite biconvex subsets is of this form, so the notion of adjacent biconvex subset generalizes the covering relation for the weak Bruhat order.

(iii) Given a real positive root  $\alpha \in \Phi_+^{\text{re}}$ , let us say that a pair  $(A, B)$  of biconvex subsets is  $\alpha$ -adjacent if  $B = A \sqcup \{\alpha\}$ . Let us say that  $(A, B)$  is  $\delta$ -adjacent if  $B = A \sqcup \mathbb{Z}_{>0}\delta$ . We conjecture that a pair  $(A, B)$  of biconvex subset is adjacent (in the sense of (ii) above) if and only if there is a root  $\beta$  (real or imaginary) such that  $(A, B)$  is  $\beta$ -adjacent. This conjecture seems reasonable in view of our current understanding of biconvex subsets, but we were not able to extract it from the papers [14, 26]. If it is correct, then Lemma 2.9 admits a converse, and “maximal totally ordered subset of  $\mathcal{V}$ ” would be a notion equivalent to that of “convex order”. In any case, Zorn’s lemma shows that any totally ordered subset of  $\mathcal{V}$  can be completed to a maximal one.

(iv) Let  $\preceq$  be a convex order. The terminal sections

$$\{\beta \in \Phi_+ \mid \beta \succ \delta\} \quad \text{and} \quad \{\beta \in \Phi_+ \mid \beta \succneq \delta\}$$

satisfy the assumptions of Lemma 2.8 (i), so there is a positive system  $X \subseteq \Phi^s$  such that  $\{\beta \in \Phi_+ \mid \beta \succ \delta\} = \{\beta \in \Phi_+ \mid \pi(\beta) \in X\}$ . This fact was announced in Section 1.7, see equation (1.1).

*Examples 2.11.* (i) Let us consider the type  $\tilde{A}_1$ . As is customary, we use  $I = \{0, 1\}$ . Then

$$\Phi_+ = \{\alpha_0 + n\delta, \alpha_1 + n\delta, (n+1)\delta \mid n \in \mathbb{N}\}.$$

There are exactly two convex orders on  $\Phi_+$ . One of them is

$$\alpha_1 \prec \alpha_1 + \delta \prec \cdots \prec \mathbb{Z}_{>0}\delta \prec \cdots \prec \alpha_0 + \delta \prec \alpha_0,$$

the other is the opposite order.

(ii) A linear form  $\theta \in (\mathbb{R}I)^*$  defines a convex preorder on  $\Phi_+$ , as follows: we say that  $\alpha \preceq \beta$  if  $\theta(\alpha)/\text{ht}(\alpha) \leq \theta(\beta)/\text{ht}(\beta)$ . For  $\theta$  general enough (outside countably many hyperplanes), this preorder is a convex order.

## 2.5 GGMS polytopes in affine type

To a compact convex subset  $K \subseteq \mathbb{R}I$ , one associates its support function  $\psi_K : (\mathbb{R}I)^* \rightarrow \mathbb{R}$ , defined by  $\psi_K(\theta) = \max(\theta(K))$ . One can reconstruct  $K$  from the datum of  $\psi_K$ . When  $K$  is a convex polytope,  $\psi_K$  is piecewise linear; the maximal regions of linearity are closed cones that cover  $(\mathbb{R}I)^*$ . These cones and their faces form the normal fan to  $P$ .

We define a GGMS polytope as a convex lattice polytope  $P \subseteq \mathbb{R}I$  whose normal fan is a coarsening of the affine Weyl fan  $\mathcal{W}$ . The latter requirement means that any face of the normal fan to  $P$  is the union of faces in  $\mathcal{W}$ , or, equivalently, that the restriction of  $\psi_P$  to each face in  $\mathcal{W}$  is linear. Since we have restricted to affine type, it suffices to check the latter condition on the open faces of  $C_T$ ,  $-C_T$ , and  $\mathfrak{t}$ .

Fix a GGMS polytope  $P$  for the rest of this section.

We begin with an observation:  $P$  has a finite number of faces of any dimension, and each of these is of the form  $P_\theta = \{x \in P \mid \langle \theta, x \rangle = \psi_P(\theta)\}$  for some  $\theta \in (\mathbb{R}I)^*$ . A vertex of  $P$  can always be written in this form, with  $\theta$  in an open cone of  $\mathcal{W}$ . Moreover, an edge of  $P$  always points in a root direction. Let us call  $X_P$  the (finite) set of all these root directions.

Let  $A \subseteq \Phi_+$  be a finite or a cofinite biconvex subset. Then we can find  $\theta$  in an open cone of  $\mathcal{W}$  such that  $A = \{\alpha \in \Phi_+ \mid \langle \theta, \alpha \rangle > 0\}$ , by Example 2.4 (ii) and Proposition 2.5. The face  $P_\theta$  is then a vertex, which we denote by  $\mu(A)$ .

Given a subset  $X \subseteq \Phi_+$ , we denote by  $\mathbb{N}X$  the  $\mathbb{N}$ -span of  $X$  in  $\mathbb{Z}I$ .

**Lemma 2.12** *Let  $A$  and  $B$  be finite or cofinite biconvex subsets. If  $A \subseteq B$ , then  $\mu(B) - \mu(A) \in \mathbb{N}(B \setminus A)$ .*

*Proof.* We choose  $\theta_0$  and  $\theta_1$  in open cones of  $\mathcal{W}$  such that  $A = \{\alpha \in \Phi_+ \mid \langle \theta_0, \alpha \rangle > 0\}$  and  $B = \{\alpha \in \Phi_+ \mid \langle \theta_1, \alpha \rangle > 0\}$ . By moving  $\theta_0$  and  $\theta_1$  if necessary, we may assume that the segment  $[\theta_0, \theta_1]$  does not meet any cone of codimension 2 in the normal fan to  $P$ . We consider  $\theta(t) = (1-t)\theta_0 + t\theta_1$ . As  $t$  varies from 0 to 1, the face  $P_{\theta(t)}$  is generally a vertex of  $P$ , but occasionally an edge; it is never a face of higher dimension. The edges and faces found in this way form a path in the 1-skeleton of  $P$  from  $\mu(A)$  to  $\mu(B)$ . The edges traversed by this path point in direction of roots  $\alpha$  such that  $\langle \theta_0, \alpha \rangle < 0 < \langle \theta_1, \alpha \rangle$ , that is, in  $B \setminus A$ . Recalling that our polytope is a lattice polytope, we conclude that  $\mu(B) - \mu(A) \in \mathbb{N}(B \setminus A)$ , as desired.  $\square$

In fact, the proof of Lemma 2.12 shows that  $\mu(A) = \mu(B)$  if  $B \setminus A$  does not meet  $X_P$ . We can then extend  $\mu$  to a map from all of  $\mathcal{V}$  to the set of vertices of  $P$  as follows. If  $A$  is a biconvex subset such that  $\delta \notin A$ , then we set  $\mu(A) = \mu(B)$ , where  $B$  is any finite biconvex subset such

that  $A \cap X_P \subseteq B \subseteq A$ ; the result does not depend on the choice of  $B$ , because the set of all possible  $B$  is filtered. Similarly, if  $A$  is a biconvex subset such that  $\delta \in A$ , then we set  $\mu(A) = \mu(B)$ , where  $B$  is any cofinite biconvex subset such that  $A \subseteq B \subseteq (A \cup (\Phi_+ \setminus X_P))$ .

Lemma 2.12 extends to any biconvex subsets  $A$  and  $B$ .

**Proposition 2.13** *Let  $A$  and  $B$  be two biconvex subsets. If  $A \subseteq B$ , then  $\mu(B) - \mu(A) \in \mathbb{N}(B \setminus A)$ . Moreover, if  $B = A \sqcup \{\alpha\}$ , where  $\alpha \in \Phi_+^{\text{re}}$ , or if  $B = A \sqcup \mathbb{Z}_{>0}\delta$ , then the segment  $[\mu(A), \mu(B)]$  is an edge of  $P$ .*

Now fix a convex order  $\preccurlyeq$  on  $\Phi_+$ . For  $\alpha \in \Phi_+$ , look at

$$A = \{\beta \in \Phi_+ \mid \beta \succ \alpha\} \quad \text{and} \quad B = \{\beta \in \Phi_+ \mid \beta \succcurlyeq \alpha\}.$$

These are biconvex subsets such that  $B = A \sqcup \{\alpha\}$ , if  $\alpha$  is real, or  $B = A \sqcup \mathbb{Z}_{>0}\delta$ , if  $\alpha$  is imaginary. By Proposition 2.13,  $\mu(B) - \mu(A) = n_\alpha \alpha$  for some non-negative integer  $n_\alpha$ . This number is nonzero only if  $\alpha \in X_P$ .

Reversing this definition, we get, for any  $A \in \mathcal{U}(\preccurlyeq)$ ,

$$\mu(A) = \begin{cases} \sum_{\alpha \in A} n_\alpha \alpha & \text{if } \delta \notin A, \\ n_\delta \delta + \sum_{\alpha \in A \cap \Phi_+^{\text{re}}} n_\alpha \alpha & \text{if } \delta \in A. \end{cases}$$

In particular, the top vertex of  $P$  is  $\mu(\Phi_+) = \sum_{\alpha \in \Phi_+} n_\alpha \alpha$ .

This collection of numbers  $(n_\alpha)$  will be called the Lusztig datum of  $P$  in direction  $\preccurlyeq$ . We will however later decorate our GGMS polytopes in order to refine the information carried by  $n_\delta$ , taking into account all the imaginary roots and their multiplicities.

### 3 Torsion pairs and Harder-Narasimhan polytopes

In this section we will study general facts about torsion pairs and Harder-Narasimhan polytopes. We consider an essentially small abelian category  $\mathcal{A}$  such that all objects have finite length. This assumption ensures that the Grothendieck group  $K_0(\mathcal{A})$  is a free abelian group, with basis the set of isomorphism class of simple objects. As usual, we denote by  $[T]$  the class in  $K_0(\mathcal{A})$  of an object  $T \in \mathcal{A}$ . Our subcategories will always be full subcategories.

### 3.1 Torsion pairs

Following [3], a torsion pair in  $\mathcal{A}$  is a pair  $(\mathcal{T}, \mathcal{F})$  of two subcategories, called the torsion class and the torsion-free class, that satisfy the following two axioms:

(T1)  $\text{Hom}_{\mathcal{A}}(X, Y) = 0$  for each  $(X, Y) \in \mathcal{T} \times \mathcal{F}$ .

(T2) Each object  $T \in \mathcal{A}$  has a subobject  $X$  such that  $(X, T/X) \in \mathcal{T} \times \mathcal{F}$ .

Axiom (T1) forces the subobject  $X$  in (T2) to be the largest subobject of  $T$  that belong to  $\mathcal{T}$ , and a fortiori to be unique; this  $X$  is called the torsion subobject of  $T$  with respect to the torsion pair  $(\mathcal{T}, \mathcal{F})$ . An equivalent set of axioms are the two requirements:

(T'1)  $\mathcal{T} = \{X \in \mathcal{A} \mid \forall Y \in \mathcal{F}, \text{Hom}(X, Y) = 0\}$ .

(T'2)  $\mathcal{F} = \{Y \in \mathcal{A} \mid \forall X \in \mathcal{T}, \text{Hom}(X, Y) = 0\}$ .

With this second formulation, it is clear that  $\mathcal{T}$  is closed under taking quotients and extensions and that  $\mathcal{F}$  is closed under taking subobjects and extensions.

Torsion pairs are ordered: we write  $(\mathcal{T}', \mathcal{F}') \preceq (\mathcal{T}'', \mathcal{F}'')$  if the following three equivalent conditions hold:

$$\mathcal{T}' \subseteq \mathcal{T}'', \quad \mathcal{F}' \supseteq \mathcal{F}'', \quad \mathcal{T}' \cap \mathcal{F}'' = \{0\}.$$

In this case, each object  $T \in \mathcal{A}$  is endowed with a three-step filtration  $0 \subseteq X' \subseteq X'' \subseteq T$ , where  $X'$  and  $X''$  are the torsion subobjects of  $T$  with respect to the torsion pair similarly decorated. Since  $\mathcal{F}'$  is stable under taking subobjects and  $\mathcal{T}''$  is stable under taking quotients, we have  $(X', X''/X', T/X'') \in (\mathcal{T}', \mathcal{F}' \cap \mathcal{T}'', \mathcal{F}'')$ .

A typical example of torsion pair is obtained by the following construction, directly translated from the well-known theories of Harder-Narasimhan filtrations and stability conditions [51, 46, 49]. Fix a group homomorphism  $\theta : K_0(\mathcal{A}) \rightarrow \mathbb{R}$  and define five subcategories  $\mathcal{I}_\theta$ ,  $\overline{\mathcal{T}}_\theta$ ,  $\mathcal{P}_\theta$ ,  $\overline{\mathcal{P}}_\theta$  and  $\mathcal{R}_\theta$  of  $\mathcal{A}$ :

- An object  $T$  is in  $\mathcal{I}_\theta$  (respectively,  $\overline{\mathcal{T}}_\theta$ ) if any nonzero quotient  $X$  of  $T$  satisfies  $\theta([X]) > 0$  (respectively,  $\theta([X]) \geq 0$ ).
- An object  $T$  is in  $\mathcal{P}_\theta$  (respectively,  $\overline{\mathcal{P}}_\theta$ ) if any nonzero subobject  $X$  of  $T$  satisfies  $\theta([X]) < 0$  (respectively,  $\theta([X]) \leq 0$ ).
- An object  $T$  is in  $\mathcal{R}_\theta$  if  $\theta([T]) = 0$  and any nonzero subobject  $X$  of  $T$  satisfies  $\theta([X]) \leq 0$ .



The objects in the category  $\mathcal{R}_\theta$  are called  $\theta$ -semistable [36]. Note that  $\mathcal{R}_\theta = \overline{\mathcal{I}}_\theta \cap \overline{\mathcal{P}}_\theta$ .

**Proposition 3.1** *Both  $(\mathcal{I}_\theta, \overline{\mathcal{P}}_\theta)$  and  $(\overline{\mathcal{I}}_\theta, \mathcal{P}_\theta)$  are torsion pairs. The category  $\mathcal{R}_\theta$  is abelian.*

*Proof.* Let us first prove that  $(\mathcal{I}_\theta, \overline{\mathcal{P}}_\theta)$  is a torsion pair. The axiom (T1) is obvious, so we have to prove the axiom (T2).

We first show that  $\mathcal{I}_\theta$  is closed under extensions. Let  $0 \rightarrow T' \rightarrow T \xrightarrow{f} T'' \rightarrow 0$  be a short exact sequence with  $T'$  and  $T''$  in  $\mathcal{I}_\theta$  and let  $g : T \rightarrow X$  be an epimorphism. The pushout of  $(f, g)$  then exhibits  $X$  as the extension of a quotient  $X''$  of  $T''$  by a quotient  $X'$  of  $T'$ . By assumption,  $\theta([X'])$  and  $\theta([X''])$  are both nonnegative, so  $\theta([X]) \geq 0$ . Moreover, equality holds only if both  $X'$  and  $X''$  are zero, thus only if  $X = 0$ .

Now let  $T \in \mathcal{A}$ . Our assumption of finite length allows us to pick a maximal element  $X$  among the subobjects of  $T$  that belong to  $\mathcal{I}_\theta$ . Suppose that  $T/X$  is not in  $\overline{\mathcal{P}}_\theta$ . Then it contains a subobject  $Y$  such that  $\theta([Y]) > 0$ , and we may assume that  $Y$  has been chosen minimal with this property. Certainly,  $Y$  does not belong to  $\mathcal{I}_\theta$ ; otherwise, the extension of  $Y$  by  $X$  inside  $T$  would belong to  $\mathcal{I}_\theta$ , contradicting the maximality of  $X$ . So  $Y$  has a nonzero quotient  $Y/Z$  such that  $\theta([Y/Z]) \leq 0$ . Since  $Z$  is a subobject of  $T/X$  properly contained in  $Y$ , the minimality of  $Y$  requires  $\theta([Z]) \leq 0$ . We thus reach a contradiction, namely  $0 \geq \theta([Z]) + \theta([Y/Z]) = \theta([Y]) > 0$ . Therefore  $T/X \in \overline{\mathcal{P}}_\theta$ , which establishes (T2).

We have thus shown that  $(\mathcal{I}_\theta, \overline{\mathcal{P}}_\theta)$  is a torsion pair. The proof for  $(\overline{\mathcal{I}}_\theta, \mathcal{P}_\theta)$  is similar. That  $\mathcal{R}_\theta$  is abelian then follows by a well-known (and easy) argument.  $\square$

Since  $(\mathcal{I}_\theta, \overline{\mathcal{P}}_\theta) \preceq (\overline{\mathcal{I}}_\theta, \mathcal{P}_\theta)$ , these two torsion pairs endow each object  $T \in \mathcal{A}$  with a three-step filtration  $0 \subseteq T_\theta^{\min} \subseteq T_\theta^{\max} \subseteq 0$ . The quotient  $T_\theta^{\max}/T_\theta^{\min}$  belongs to  $\mathcal{R}_\theta = \overline{\mathcal{I}}_\theta \cap \overline{\mathcal{P}}_\theta$ .

**Proposition 3.2** *Let  $\theta : K_0(\mathcal{A}) \rightarrow \mathbb{R}$  be a group homomorphism and let  $T \in \mathcal{A}$ . Then*

$$\theta([T_\theta^{\min}]) = \theta([T_\theta^{\max}]) \geq \theta([X])$$

*for any subobject  $X \subseteq T$ . Equality holds if and only if  $T_\theta^{\min} \subseteq X \subseteq T_\theta^{\max}$  and  $X/T_\theta^{\min}$  is  $\theta$ -semistable.*

*Proof.* We adopt the notation of the statement. Let  $X$  be a subobject of  $T$ . Since  $T_\theta^{\min} \in \mathcal{I}_\theta$ , we have  $\theta([T_\theta^{\min}/(X \cap T_\theta^{\min})]) \geq 0$ , with equality only if  $T_\theta^{\min} \subseteq X$ . Since  $T/T_\theta^{\max} \in \mathcal{P}_\theta$ , we have  $\theta([(X + T_\theta^{\max})/T_\theta^{\max}]) \leq 0$ , with equality only if  $X \subseteq T_\theta^{\max}$ . Lastly, we note that  $(X \cap T_\theta^{\max})/(X \cap T_\theta^{\min})$  is a subobject of  $T_\theta^{\max}/T_\theta^{\min}$ ; since the latter is in  $\overline{\mathcal{P}}_\theta$ , we have  $\theta([(X \cap T_\theta^{\max})/(X \cap T_\theta^{\min})]) \leq 0$ , with equality if and only if  $(X \cap T_\theta^{\max})/(X \cap T_\theta^{\min})$  is  $\theta$ -semistable. The result now follows from the Grassmann relation  $[X + T_\theta^{\max}] + [X \cap T_\theta^{\min}] = [X] + [T_\theta^{\max}]$ .  $\square$

### 3.2 Harder-Narasimhan polytopes

We set  $K_0(\mathcal{A})_{\mathbb{R}} = K_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ . We view this  $\mathbb{R}$ -vector space as the inductive limit of its finite dimensional subspaces; it is thus a locally convex topological vector space. Linear forms on this vector space are automatically continuous. We identify the dual space of  $K_0(\mathcal{A})_{\mathbb{R}}$  with the homomorphism group  $\text{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), \mathbb{R})$ .

Given an object  $T \in \mathcal{A}$ , there are finitely many classes  $[X]$  of subobjects  $X \subseteq T$ . The convex hull in  $K_0(\mathcal{A})_{\mathbb{R}}$  of all these points is a convex lattice polytope. We call it the Harder-Narasimhan polytope of  $T$  and we denote it by  $\text{Pol}(T)$ . The support function  $\psi_{\text{Pol}(T)}$  of this polytope is defined as the function on the dual of  $K_0(\mathcal{A})_{\mathbb{R}}$  that maps a linear form  $\theta : K_0(\mathcal{A})_{\mathbb{R}} \rightarrow \mathbb{R}$  to its maximum on  $\text{Pol}(T)$ . As in section 2.5,  $P_{\theta} = \{x \in \text{Pol}(T) \mid \theta(x) = \psi_{\text{Pol}(T)}(\theta)\}$  is a face of  $\text{Pol}(T)$ .

The inclusion  $i : \mathcal{R}_{\theta} \subseteq \mathcal{A}$  is an exact functor, so it induces a group homomorphism  $K_0(i) : K_0(\mathcal{R}_{\theta}) \rightarrow K_0(\mathcal{A})$  and a corresponding linear map  $K_0(i)_{\mathbb{R}}$ . Proposition 3.2 then receives the following interpretation.

**Corollary 3.3** *Let  $\theta \in \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), \mathbb{R})$ . Let us denote by  $Q \subseteq K_0(\mathcal{R}_{\theta})_{\mathbb{R}}$  the HN polytope of  $T_{\theta}^{\max}/T_{\theta}^{\min}$ , regarded as an object of  $\mathcal{R}_{\theta}$ . Then*

$$P_{\theta} = [T_{\theta}^{\min}] + K_0(i)_{\mathbb{R}}(Q).$$

Thus the face  $P_{\theta}$  of  $\text{Pol}(T)$  is the HN polytope of  $T_{\theta}^{\max}/T_{\theta}^{\min}$ , computed relative to the category  $\mathcal{R}_{\theta}$ , and shifted by  $[T_{\theta}^{\min}]$ .

A consequence of this observation is that  $[T_{\theta}^{\min}]$  and  $[T_{\theta}^{\max}]$  are vertices of  $\text{Pol}(T)$ . Another noteworthy consequence is the following rigidity property: if  $x$  is a vertex of  $\text{Pol}(T)$ , then  $T$  has a unique subobject  $X$  such that  $[X] = x$ .

We may also interpret the categories  $\mathcal{I}_{\theta}$ ,  $\overline{\mathcal{I}}_{\theta}$ , etc., in terms of HN polytopes. For instance, an object  $T$  belongs to  $\overline{\mathcal{I}}_{\theta}$  if and only if  $T = T_{\theta}^{\max}$ , hence if and only if the top vertex  $[T]$  of  $\text{Pol}(T)$  lies on the face defined by  $\theta$ .

Given  $T$  and  $\theta$ , the subfaces of  $P_{\theta}$  are obtained by perturbing slightly  $\theta$ . The following result states this formally.

**Proposition 3.4** *Let  $(\theta, \eta) \in \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), \mathbb{R})^2$ , let  $T \in \mathcal{A}$ , let  $X = T_{\theta}^{\max}/T_{\theta}^{\min}$ , and let  $i : \mathcal{R}_{\theta} \subseteq \mathcal{A}$  be the inclusion functor. Let  $0 \subseteq X_{\eta \circ K_0(i)}^{\min} \subseteq X_{\eta \circ K_0(i)}^{\max} \subseteq X$  be the filtration of  $X$ , regarded as an object in  $\mathcal{R}_{\theta}$ , relative to the group homomorphism  $\eta \circ K_0(i) \in \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{R}_{\theta}), \mathbb{R})$ . Call  $T_{\theta}^{\min} \subseteq T' \subseteq T'' \subseteq T_{\theta}^{\max}$  the pull-back of this filtration by the canonical epimorphism  $T_{\theta}^{\max} \rightarrow X$ . Then for  $m$  large enough,  $T' = T_{m\theta+\eta}^{\min}$  and  $T'' = T_{m\theta+\eta}^{\max}$ .*

*Proof.* The classes in  $K_0(\mathcal{A})$  of subquotients of  $T$  are finitely many. Pick  $m$  large enough so that, for all subquotients  $Z$  of  $T$ ,

$$\theta([Z]) > 0 \implies (m\theta + \eta)([Y]) > 0 \quad \text{and} \quad \theta([Z]) < 0 \implies (m\theta + \eta)([Y]) < 0.$$

Each nonzero quotient  $Y$  of  $T_\theta^{\min}$  satisfies  $\theta([Y]) > 0$ , hence satisfies  $(m\theta + \eta)([Y]) > 0$ . Therefore  $T_\theta^{\min} \in \mathcal{I}_{m\theta+\eta}$ , and so  $T_\theta^{\min} \subseteq T_{m\theta+\eta}^{\min}$ . The quotient  $U = T_{m\theta+\eta}^{\min}/T_\theta^{\min}$  belongs to  $\mathcal{I}_{m\theta+\eta}$  and to  $\overline{\mathcal{P}}_\theta$ , so we have  $(m\theta + \eta)([U]) \geq 0$  and  $\theta([U]) \leq 0$ , which forces  $\theta([U]) = 0$ .

In a similar fashion, we see that  $T_{m\theta+\eta}^{\max} \subseteq T_\theta^{\max}$  and  $\theta([T_\theta^{\max}/T_{m\theta+\eta}^{\max}]) = 0$ . We conclude that

$$T_\theta^{\min} \subseteq T_{m\theta+\eta}^{\min} \subseteq T_{m\theta+\eta}^{\max} \subseteq T_\theta^{\max}$$

and that the subquotients of this filtration belong to  $\mathcal{R}_\theta$ .

Reducing modulo  $T_\theta^{\min}$ , we get a three-step filtration  $0 \subseteq X' \subseteq X'' \subseteq X$  of  $X = T_\theta^{\max}/T_\theta^{\min}$ , viewed as an object of  $\mathcal{R}_\theta$ . Any nonzero quotient  $Y$  of  $X'$  in  $\mathcal{R}_\theta$  is a nonzero quotient of  $T_{m\theta+\eta}^{\min}$  in  $\mathcal{A}$  such that  $\theta([Y]) = 0$ , and so  $\eta([Y]) = (m\theta + \eta)([Y]) > 0$ . Therefore  $X'$  belongs to the subcategory  $\mathcal{I}_{\eta \circ K_0(i)}$  of  $\mathcal{R}_\theta$ . One checks in a similar fashion that  $X''/X' \in \mathcal{R}_{\eta \circ K_0(i)}$  and  $X/X'' \in \mathcal{P}_{\eta \circ K_0(i)}$ . We conclude that  $X' = X_{\eta \circ K_0(i)}^{\min}$  and  $X'' = X_{\eta \circ K_0(i)}^{\max}$ .  $\square$

Let us now compare this construction to the more usual notion of Harder-Narasimhan filtration. To define the latter, we need to fix a pair  $(\eta, \theta) \in \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), \mathbb{R})^2$  such that  $\eta([T]) > 0$  for each nonzero object  $T$ . The slope of a nonzero object  $T \in \mathcal{A}$  is defined as  $\mu(T) = \theta([T])/\eta([T])$  and an object  $T$  is called semistable if it is zero or if  $\mu(X) \leq \mu(T)$  for any nonzero subobject  $X \subseteq T$ . It can then be shown that any object  $T \in \mathcal{A}$  has a finite filtration

$$0 = T_0 \subset T_1 \subset \cdots \subset T_{\ell-1} \subset T_\ell = T \tag{3.1}$$

whose subquotients are nonzero and semistable, with moreover  $\mu(T_k/T_{k-1})$  decreasing with  $k$  (see for instance [46], close to the present context). This filtration is unique and is called the Harder-Narasimhan filtration of  $T$ .

Given  $a \in \mathbb{R}$ , the torsion subobject of  $T$  with respect to the torsion pair  $(\mathcal{I}_{\theta-a\eta}, \overline{\mathcal{P}}_{\theta-a\eta})$  is  $T_k$ , where  $k$  is the largest index such that  $\mu(T_k/T_{k-1}) > a$ . In our former notation, this means that  $T_k = T_{\theta-a\eta}^{\min}$ ; in particular,  $[T_k]$  is a vertex of  $\text{Pol}(T)$ . One can be even more precise: the linear map  $\varphi : K_0(\mathcal{A})_{\mathbb{R}} \rightarrow \mathbb{R}^2$  given in coordinates as  $(\eta, \theta)$  projects  $\text{Pol}(T)$  to a convex polygon of the plane, and the upper ridge of this polygon is the polygonal line going successively through the points  $\varphi([T_k])$ , for  $0 \leq k \leq \ell$ . We leave the proof of this fact to the reader.

The polygonal line just obtained is what Shatz calls the HN polygon [51]. Thus our HN polytopes are a multidimensional analog of those HN polygons; they simply take into account

the existence of a whole space of stability conditions. There may well exist sensible adaptations of this notion of HN polytope to other contexts where spaces of stability conditions have been defined (see for instance [12]).

- Remarks 3.5.* (i) In Corollary 3.3, the map  $K_0(i)_{\mathbb{R}}$  induces a real loss of information. For example, in our study of vertical edges (Sections 1.5 and 7.5), the category  $\mathcal{R}_{\theta}$  has infinitely many simple objects; since they have the same dimension-vectors, their classes have the same image by  $K_0(i)_{\mathbb{R}}$ .
- (ii) The HN polytope of the direct sum of two objects is the Minkowski sum of the HN polytopes of the two objects.

### 3.3 Nested families of torsion pairs

The subobjects of  $T$  that appear in the HN filtration (3.1) are the torsion subobjects with respect to the torsion pairs  $(\mathcal{I}_{\theta+a\eta}, \overline{\mathcal{P}}_{\theta+a\eta})$ , as  $a$  varies over  $\mathbb{R}$ . Observe that, in the notation of Section 3.1,

$$\forall (a, b) \in \mathbb{R}^2, \quad a \leq b \implies (\mathcal{I}_{\theta+a\eta}, \overline{\mathcal{P}}_{\theta+a\eta}) \preceq (\mathcal{I}_{\theta+b\eta}, \overline{\mathcal{P}}_{\theta+b\eta}).$$

This prompts the following definition: a nested family of torsion pairs is the datum of a family  $(\mathcal{T}_a, \mathcal{F}_a)_{a \in A}$  of torsion pairs, indexed by a totally ordered set  $A$ , such that

$$\forall (a, b) \in A^2, \quad a \leq b \implies (\mathcal{T}_a, \mathcal{F}_a) \preceq (\mathcal{T}_b, \mathcal{F}_b).$$

This definition is certainly less general than Rudakov's study [49] but is sufficient for our purposes.

A nested family of torsion pairs  $(\mathcal{T}_a, \mathcal{F}_a)_{a \in A}$  in  $\mathcal{A}$  induces a non-decreasing filtration  $(T_a)_{a \in A}$  on any object  $T \in \mathcal{A}$ : simply define  $T_a$  as the torsion subobject of  $T$  with respect to  $(\mathcal{T}_a, \mathcal{F}_a)$ . As already shown in Section 3.1, the object  $T_b/T_a$  is in  $\mathcal{F}_a \cap \mathcal{T}_b$  whenever  $a \leq b$ .

## 4 Background on preprojective algebras

### 4.1 Basic definitions

We fix a base field  $K$ , which we assume for convenience to be algebraically closed of characteristic 0. As in Section 2.1, we fix a graph  $(I, E)$ , where  $I$  is the set of vertices and  $E$  the set

of edges. We denote by  $H$  the set of oriented edges of this graph. Thus each edge in  $E$  gives birth to two oriented edges in  $H$ , and  $H$  comes with a source map  $s : H \rightarrow I$ , a target map  $t : H \rightarrow I$  and a fixed-point free involution  $*$  such that  $s(a) = t(a^*)$  for each  $a \in H$ .

An orientation is a subset  $\Omega \subset H$  such that  $H = \Omega \sqcup \Omega^*$ . Such an orientation yields a quiver  $Q = (I, \Omega, s, t)$ , and then  $\overline{Q} = (I, H, s, t)$  is the double quiver of  $Q$ . We set  $\varepsilon(a) = 1$  if  $a \in \Omega$  and  $\varepsilon(a) = -1$  if  $a \notin \Omega$ .

Let  $K\overline{Q}$  be the path algebra of  $\overline{Q}$ . The linear span  $\mathbf{S} = \text{span}_K(e_i)_{i \in I}$  of the lazy paths is a commutative semisimple subalgebra of  $K\overline{Q}$ . The linear span  $\mathbf{A} = \text{span}_K(a)_{a \in H}$  of the paths of length one is an  $\mathbf{S}$ - $\mathbf{S}$ -bimodule. Then  $K\overline{Q}$  is the tensor algebra  $T_{\mathbf{S}}\mathbf{A}$ . For  $i \in I$ , set

$$\rho_i = \sum_{\substack{a \in H \\ s(a)=i}} \varepsilon(a) a^* a,$$

the so-called preprojective relation at vertex  $i$ . The linear span  $\mathbf{R} = \text{span}_K(\rho_i)_{i \in I}$  is a  $\mathbf{S}$ - $\mathbf{S}$ -subbimodule of  $K\overline{Q}$ .

By definition, the preprojective algebra of  $Q$  is the quotient of  $K\overline{Q}$  by the ideal generated by  $\mathbf{R}$ . This is an augmented algebra over  $\mathbf{S}$ . Its completion with respect to the augmentation ideal is called the completed preprojective algebra and is denoted by  $\Lambda_Q$ . For brevity, we will generally drop the  $Q$  in the notation  $\Lambda_Q$ . Completing has the effect that the augmentation ideal becomes the Jacobson radical; thus the simple  $\Lambda$ -modules are just the simple  $\mathbf{S}$ -modules, namely the one dimensional modules  $S_i$ . We denote by  $\Lambda\text{-mod}$  the category of finite dimensional  $\Lambda$ -modules.

The involution  $*$  on the set  $E$  of oriented edges induces an anti-automorphism of  $\Lambda$ . If  $M$  is a finite dimensional  $\Lambda$ -module, then we denote by  $M^*$  the dual module  $\text{Hom}_K(M, K)$ , viewed as a left module by means of this anti-automorphism.

Occasionally, we will have to write  $\Lambda$ -modules in a concrete fashion. Our notation is as follows: a  $\Lambda$ -module  $M$  is an  $I$ -graded vector space  $M = \bigoplus_{i \in I} M_i$ ; an arrow  $a \in H$  acts on  $M$  by a linear map  $M_a : M_{s(a)} \rightarrow M_{t(a)}$ .

Since the simple  $\Lambda$ -modules are the modules  $S_i$ , concentrated at a vertex of the quiver, it is natural to present a special notation designed to analyze a  $\Lambda$ -module  $M$  locally around a vertex  $i$ . Specifically, we break the datum of  $M$  in two parts: the first part consists of the vector spaces  $M_j$  for  $j \neq i$  and of the linear maps between them; the second part consists of the vector spaces and of the linear maps that appear in the diagram

$$\bigoplus_{\substack{a \in H \\ s(a)=i}} M_{t(a)} \xrightarrow{(M_a^*)} M_i \xrightarrow{(\varepsilon(a)M_a)} \bigoplus_{\substack{a \in H \\ s(a)=i}} M_{t(a)}.$$

For brevity, we will write the latter as

$$\widetilde{M}_i \xrightarrow{M_{\text{in}(i)}} M_i \xrightarrow{M_{\text{out}(i)}} \widetilde{M}_i. \quad (4.1)$$

With this notation, the preprojective relation at  $i$  is  $M_{\text{in}(i)}M_{\text{out}(i)} = 0$ .

We define the dimension vector of  $M$  to be  $\underline{\dim} M = \sum_{i \in I} (\dim M_i) \alpha_i$ . The dimension-vector gives an isomorphism between the Grothendieck group  $K_0(\Lambda\text{-mod})$  and the root lattice  $\mathbb{Z}I$ . Crawley-Boevey's formula (Lemma 1 in [16]) gives a module-theoretic meaning to the bilinear form on the root lattice:

$$\dim \text{Hom}_\Lambda(M, N) + \dim \text{Hom}_\Lambda(N, M) - \dim \text{Ext}_\Lambda^1(M, N) = (\underline{\dim} M, \underline{\dim} N) \quad (4.2)$$

for any finite dimensional  $\Lambda$ -modules  $M$  and  $N$ .

*Remark 4.1.* The HN polytope  $\text{Pol}(T)$  of an object  $T \in \Lambda\text{-mod}$  lives in  $K_0(\Lambda\text{-mod})_{\mathbb{R}} \cong \mathbb{R}I$ . Since the duality exchanges submodules and quotients and leaves the dimension-vector unchanged,  $\text{Pol}(T^*)$  is the image of  $\text{Pol}(T)$  under the involution  $x \mapsto \underline{\dim} T - x$  of  $\mathbb{R}I$ . We leave it to the reader to check the equalities  $\mathcal{J}_{-\theta} = (\mathcal{P}_\theta)^*$ ,  $\overline{\mathcal{J}}_{-\theta} = (\overline{\mathcal{P}}_\theta)^*$  and  $\mathcal{R}_{-\theta} = (\mathcal{R}_\theta)^*$ , for any  $\theta \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}I, \mathbb{R})$ .

## 4.2 Projective resolutions

We now recall Geiß, Leclerc and Schröer's description of the extension groups in the category  $\Lambda\text{-mod}$  (see [23], Section 8).

Consider the complex of  $\Lambda$ -bimodules

$$\Lambda \otimes_{\mathbf{S}} \mathbf{R} \otimes_{\mathbf{S}} \Lambda \xrightarrow{d_1} \Lambda \otimes_{\mathbf{S}} \mathbf{A} \otimes_{\mathbf{S}} \Lambda \xrightarrow{d_0} \Lambda \otimes_{\mathbf{S}} \mathbf{S} \otimes_{\mathbf{S}} \Lambda \rightarrow \Lambda \rightarrow 0, \quad (4.3)$$

where the map on the right is multiplication in  $\Lambda$ , where for each  $a \in H$

$$d_0(1 \otimes a \otimes 1) = a \otimes e_{s(a)} \otimes 1 - 1 \otimes e_{t(a)} \otimes a,$$

and where for each  $i \in I$

$$d_1(1 \otimes \rho_i \otimes 1) = \sum_{\substack{a \in H \\ s(a)=i}} \varepsilon(a)(a^* \otimes a \otimes 1 + 1 \otimes a^* \otimes a).$$

Then (4.3) is the beginning of a projective resolution of  $\Lambda$ , by [23], Lemma 8.1.1.

Given  $M, N \in \Lambda\text{-mod}$ , one can apply  $\text{Hom}_\Lambda(? \otimes_\Lambda M, N)$  to (4.3). One then obtains the complex

$$0 \rightarrow \bigoplus_{i \in I} \text{Hom}_K(M_i, N_i) \xrightarrow{d_{M,N}^0} \bigoplus_{a \in H} \text{Hom}_K(M_{s(a)}, N_{t(a)}) \xrightarrow{d_{M,N}^1} \bigoplus_{i \in I} \text{Hom}_K(M_i, N_i), \quad (4.4)$$

where

$$d_{M,N}^0 : (f_i)_{i \in I} \mapsto (N_a f_{s(a)} - f_{t(a)} M_a)_{a \in H}$$

and

$$d_{M,N}^1 : (g_a)_{a \in H} \mapsto \left( \sum_{\substack{a \in H \\ s(a)=i}} \varepsilon(a) (N_a^* g_a + g_a^* M_a) \right)_{i \in I}.$$

Thus for  $k \in \{0, 1\}$ , the extension group  $\text{Ext}_\Lambda^k(M, N)$  can be identified to the cohomology groups in degree  $k$  of the complex (4.4).

In [23], Section 8.2, Geiß, Leclerc and Schröer explain that in this identification, the bilinear map

$$\tau_1 : \left( \bigoplus_{a \in H} \text{Hom}_K(M_{s(a)}, N_{t(a)}) \right) \times \left( \bigoplus_{a \in H} \text{Hom}_K(N_{s(a)}, M_{t(a)}) \right) \rightarrow K$$

defined by

$$\tau_1((g_a), (h_a)) = \sum_{i \in I} \text{Tr} \left( \sum_{\substack{a \in H \\ s(a)=i}} \varepsilon(a) g_a^* h_a \right)$$

induces a non-degenerate pairing between  $\text{Ext}_\Lambda^1(M, N)$  and  $\text{Ext}_\Lambda^1(N, M)$ . Note that because of the cyclicity of the trace and of the presence of the signs  $\varepsilon(a)$ , this pairing  $\tau_1$  is antisymmetric.

One sees likewise that the bilinear map

$$\tau_2 : \left( \bigoplus_{i \in I} \text{Hom}_K(M_i, N_i) \right) \times \left( \bigoplus_{i \in I} \text{Hom}_K(N_i, M_i) \right) \rightarrow K$$

defined by

$$\tau_2((f_i), (h_i)) = \sum_{i \in I} \text{Tr}(f_i h_i)$$

induces a non-degenerate pairing between  $\text{coker } d_{M,N}^1$  and  $\text{Hom}_\Lambda(N, M)$ .

### 4.3 Lusztig's nilpotent varieties

Given a dimension-vector  $\nu \in \mathbb{N}I$ , we can form the  $I$ -graded vector space  $\bigoplus_{i \in I} K^{\nu_i}$ . A structure of  $\Lambda$ -module on this vector space is then specified by linear maps  $T_a : K^{\nu_{s(a)}} \rightarrow K^{\nu_{t(a)}}$ , for each  $a \in H$ . A family  $(T_a)_{a \in H}$  of such linear maps forms a point in

$$\text{Rep}_K(\overline{Q}, \nu) = \bigoplus_{a \in H} \text{Hom}_K(K^{\nu_{s(a)}}, K^{\nu_{t(a)}}).$$

In order that the action of these linear maps  $T_a$  give an action of the completed preprojective algebra, one must impose the preprojective relations and the nilpotency condition. These equations define a subvariety

$$\Lambda(\nu) \subseteq \text{Rep}_K(\overline{Q}, \nu),$$

called the affine variety of representations of  $\Lambda$  or Lusztig's nilpotent variety. In the sequel, we will usually denote an element of  $\Lambda(\nu)$  by simply  $T$  instead of  $(T_a)$ , tacitly agreeing that we use  $T_i = K^{\nu_i}$  to get the full datum necessary to define a  $\Lambda$ -module.

The group  $G(\nu) = \prod_{i \in I} \text{GL}_{\nu_i}(K)$  acts by conjugation on  $\text{Rep}_K(\overline{Q}, \nu)$ . This action preserves the nilpotent variety  $\Lambda(\nu)$ . The isomorphism class of a  $\Lambda$ -module of dimension-vector  $\nu$  can then be regarded as a  $G(\nu)$ -orbit in  $\Lambda(\nu)$ .

In [39], Lusztig shows that  $\Lambda(\nu)$  is a Lagrangian subvariety of  $\text{Rep}_K(\overline{Q}, \nu)$ ; in particular, all the irreducible components of  $\Lambda(\nu)$  have dimension  $\dim(\text{Rep}_K(\overline{Q}, \nu))/2$ . A straightforward calculation shows that this common dimension is

$$\dim \Lambda(\nu) = \dim G(\nu) - (\nu, \nu)/2. \quad (4.5)$$

As in the introduction, we denote by  $\mathfrak{B}(\nu) = \text{Irr } \Lambda(\nu)$  the set of irreducible components of the nilpotent variety and we define  $\mathfrak{B} = \bigsqcup_{\nu \in \mathbb{N}I} \mathfrak{B}(\nu)$ . This set  $\mathfrak{B}$  is endowed with the structure of a crystal, defined by Lusztig ([38], Section 8), which we quickly recall.

The weight of an element  $Z \in \mathfrak{B}(\nu)$  is  $\text{wt } Z = \nu$ . The number  $\varphi_i(Z)$  is the dimension of the  $i$ -head of a general point  $T \in Z$ . The number  $\varepsilon_i(Z)$  can then be found by the general formula  $\varphi_i(Z) - \varepsilon_i(Z) = \langle \alpha_i^\vee, \text{wt } Z \rangle$ . The operators  $\tilde{e}_i$  and  $\tilde{f}_i$  generically add and remove a copy of  $S_i$  at the top of a module  $T \in Z$ . In other words, the relationship  $Z' = \tilde{e}_i Z$  corresponds to extensions  $0 \rightarrow T \rightarrow T' \rightarrow S_i \rightarrow 0$  as general as possible: if  $T$  runs over a dense open subset of  $Z$ , then  $T'$  will also run over a dense open subset of  $Z'$ , and vice versa. The duality  $*$  corresponds to the involution  $Z \mapsto Z^*$  on  $\mathfrak{B}$ , which preserves the weight. We refer the reader to the literature for the formal definitions.

Kashiwara and Saito show in [34] that the crystal  $\mathfrak{B}$  is isomorphic to the crystal  $B(-\infty)$  of  $U_q(\mathfrak{n}_+)$ . This isomorphism is canonical, because the only endomorphism of the crystal  $B(-\infty)$  is the identity.



#### 4.4 The canonical decomposition of a component

In this section, we quickly recall Crawley-Boevey and Schröer's results on canonical decomposition of irreducible components of module varieties [17], specializing their results to the case of nilpotent varieties.

Let  $\nu'$  and  $\nu''$  be two dimension-vectors. The function  $(T', T'') \mapsto \dim \operatorname{Ext}_{\Lambda}^1(T', T'')$  on  $\Lambda(\nu') \times \Lambda(\nu'')$  is upper semicontinuous. Given  $Z' \in \operatorname{Irr} \Lambda(\nu')$  and  $Z'' \in \operatorname{Irr} \Lambda(\nu'')$ , we denote its minimum on  $Z' \times Z''$  by  $\operatorname{ext}_{\Lambda}^1(Z', Z'')$ . Then  $\operatorname{ext}_{\Lambda}^1(Z', Z'') = \dim \operatorname{Ext}_{\Lambda}^1(T', T'')$  for  $(T', T'')$  general in  $Z' \times Z''$ .

For  $1 \leq i \leq n$ , let  $\nu_i$  be dimension-vectors and let  $Z_i \in \operatorname{Irr} \Lambda(\nu_i)$ . Set  $\nu = \nu_1 + \cdots + \nu_n$  and denote by  $Z_1 \oplus \cdots \oplus Z_n$  the set of all modules in  $\Lambda(\nu)$  that are isomorphic to a direct sum  $T_1 \oplus \cdots \oplus T_n$ , with  $T_i \in Z_i$  for all  $1 \leq k \leq n$ . This is an irreducible subset by  $\Lambda(\nu)$ . Its closure  $\overline{Z_1 \oplus \cdots \oplus Z_n}$  is an irreducible component of  $\Lambda(\nu)$  if and only if  $\operatorname{ext}_{\Lambda}^1(Z_i, Z_j) = 0$  for all  $i \neq j$ .

Conversely, for any  $Z \in \operatorname{Irr} \Lambda(\nu)$ , there exists  $n$ ,  $\nu_i$  and  $Z_i$  as above such that the generic point in  $Z_i$  is an indecomposable  $\Lambda$ -module and

$$Z = \overline{Z_1 \oplus \cdots \oplus Z_n}.$$

Furthermore, the  $Z_i$  are unique up to permutation. This is called the canonical decomposition of  $Z$ .

#### 4.5 Torsion pairs in $\Lambda$ -mod

Here we consider the constructions of Section 3 in the category  $\Lambda$ -mod. Since we are primarily interested in the crystal  $B(-\infty)$ , we need to make sure that our constructions go down to the level of irreducible components of the nilpotent varieties.

**Proposition 4.2** *Let  $\nu \in \mathbb{N}I$  be a dimension-vector and let  $\theta : \mathbb{R}I \rightarrow \mathbb{R}$  be a linear form.*

- (i) *For each  $\xi \in \mathbb{N}I$ , the set of all  $T \in \Lambda(\nu)$  that contain a submodule of dimension-vector  $\xi$  is closed.*
- (ii) *There are finitely many polytopes  $\operatorname{Pol}(T)$ , for  $T \in \Lambda(\nu)$ . For each polytope  $P \subseteq \mathbb{R}I$ , the set  $\{T \in \Lambda(\nu) \mid \operatorname{Pol}(T) = P\}$  is constructible.*
- (iii) *For each category  $\mathcal{C}$  among  $\mathcal{I}_{\theta}$ ,  $\overline{\mathcal{I}}_{\theta}$ ,  $\mathcal{P}_{\theta}$ ,  $\overline{\mathcal{P}}_{\theta}$  and  $\mathcal{R}_{\theta}$ , the subset  $\{T \in \Lambda(\nu) \mid T \in \mathcal{C}\}$  is open in  $\Lambda(\nu)$ .*

*Proof.* When we view a point  $T \in \Lambda(\nu)$  as a  $\Lambda$ -module, we tacitly agree that the underlying  $I$ -graded vector space of this module is  $\bigoplus_{i \in I} K^{\nu_i}$ . Let  $X$  be the set of all its  $I$ -graded vector subspaces  $\bigoplus_{i \in I} V_i$  of dimension-vector  $\xi$ ; as a product of Grassmannians,  $X$  is naturally endowed with the structure of a smooth projective variety. The incidence variety  $Y$  consisting of all pairs  $(T, V) \in \Lambda(\nu) \times X$  such that  $T_a(V_{s(a)}) \subseteq V_{t(a)}$  for all  $a \in H$  is closed. The first projection  $Y \rightarrow \Lambda(\nu)$  is therefore a projective morphism, hence is proper. Its image is therefore closed, which shows (i).

Let  $R = (\mathbb{N}I) \cap (\nu - \mathbb{N}I)$ ; this is a finite set. If  $T \in \Lambda(\nu)$ , then the dimension-vectors of the submodules of  $T$  form a subset  $S(T)$  of  $R$ . Assertion (i) says that  $\{T \in \Lambda(\nu) \mid \xi \in S(T)\}$  is closed for each  $\xi \in R$ . This implies that  $\{T \in \Lambda(\nu) \mid S(T) = S\}$  is locally closed for each subset  $S \subseteq R$ . Gathering these locally closed subsets according to the convex hull of  $S$ , we obtain assertion (ii).

According to a remark following Corollary 3.3, a point  $T \in \Lambda(\nu)$  belongs to  $\overline{\mathcal{T}}_\theta$  if and only if  $\nu$  lies on the face of  $\text{Pol}(T)$  defined by  $\theta$ . This condition means that  $S(T)$  does not meet  $\{\xi \in R \mid \langle \theta, \xi \rangle > \langle \theta, \nu \rangle\}$ . Thus assertion (i) exhibits  $\{T \in \Lambda(\nu) \mid T \in \overline{\mathcal{T}}_\theta\}$  as a finite intersection of open subsets of  $\Lambda(\nu)$ . This shows the case  $\mathcal{C} = \overline{\mathcal{T}}_\theta$  in assertion (iii). The other cases are dealt with in a similar fashion.  $\square$

Now let us fix a torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\Lambda\text{-mod}$ . All torsion submodules mentioned hereafter in this section are taken with respect to it. We make the following assumption:

(O) For each  $\nu \in \mathbb{N}I$ , both sets  $\{T \in \Lambda(\nu) \mid T \in \mathcal{T}\}$  and  $\{T \in \Lambda(\nu) \mid T \in \mathcal{F}\}$  are open.

Under this assumption, it is legitimate to consider the set  $\mathfrak{T}(\nu)$  (respectively,  $\mathfrak{F}(\nu)$ ) of all irreducible components of  $\Lambda(\nu)$  whose general point belongs to  $\mathcal{T}$  (respectively,  $\mathcal{F}$ ). We then get two subsets

$$\mathfrak{T} = \bigsqcup_{\nu \in \mathbb{N}I} \mathfrak{T}(\nu) \quad \text{and} \quad \mathfrak{F} = \bigsqcup_{\nu \in \mathbb{N}I} \mathfrak{F}(\nu)$$

of  $\mathfrak{B}$ . Our aim now is to construct a bijection  $\Xi : \mathfrak{T} \times \mathfrak{F} \rightarrow \mathfrak{B}$  that reflects at the component level the decomposition of  $\Lambda$ -modules.

Let  $(\nu_t, \nu_f) \in (\mathbb{N}I)^2$ . We set  $\nu = \nu_t + \nu_f$  and define  $\Lambda^{\mathcal{T}}(\nu_t) = \{T_t \in \Lambda(\nu_t) \mid T_t \in \mathcal{T}\}$  and  $\Lambda^{\mathcal{F}}(\nu_f) = \{T_f \in \Lambda(\nu_f) \mid T_f \in \mathcal{F}\}$ . We define  $\Theta(\nu_t, \nu_f)$  as the set of all tuples  $(T, T_t, T_f, f, g)$  such that  $T \in \Lambda(\nu)$ ,  $(T_t, T_f) \in \Lambda^{\mathcal{T}}(\nu_t) \times \Lambda^{\mathcal{F}}(\nu_f)$ , and  $0 \rightarrow T_t \xrightarrow{f} T \xrightarrow{g} T_f \rightarrow 0$  is an exact sequence in  $\Lambda\text{-mod}$ . This is a quasi-affine algebraic variety. We can then form the diagram

$$\Lambda^{\mathcal{T}}(\nu_t) \times \Lambda^{\mathcal{F}}(\nu_f) \xleftarrow{p} \Theta(\nu_t, \nu_f) \xrightarrow{q} \Lambda(\nu) \tag{4.6}$$

in which  $p$  and  $q$  are the obvious projections.

**Lemma 4.3** *The map  $p$  is a locally trivial fibration with a smooth and connected fiber of dimension  $\dim G(\nu) - (\nu_t, \nu_f)$ . The image of  $q$  is the set of all points  $T \in \Lambda(\nu)$  whose torsion submodule has dimension-vector  $\nu_t$ . The non-empty fibers of  $q$  are isomorphic to  $G(\nu_t) \times G(\nu_f)$ .*

*Proof.* The statements concerning  $q$  are obvious, so we only have to deal with  $p$ .

The points  $(T_a)$ ,  $(T_{t,a})$  and  $(T_{f,a})$ , chosen in the nilpotent varieties  $\Lambda(\nu)$ ,  $\Lambda^{\mathcal{T}}(\nu_t)$  and  $\Lambda^{\mathcal{F}}(\nu_f)$ , define  $\Lambda$ -module structures on the  $I$ -graded vector spaces  $T_i = K^{\nu_i}$ ,  $T_{t,i} = K^{\nu_{t,i}}$  and  $T_{f,i} = K^{\nu_{f,i}}$ .

Let first consider the complex (4.4) from Section 4.2, with the  $\Lambda$ -modules  $T_f$  and  $T_t$  in place of  $M$  and  $N$ . The maps  $d_{T_f, T_t}^0$  and  $d_{T_f, T_t}^1$  of the complex depend on the datum of the arrows  $(T_{t,a}) \in \Lambda^{\mathcal{T}}(\nu_t)$  and  $(T_{f,a}) \in \Lambda^{\mathcal{F}}(\nu_f)$ , but the spaces of the complex depend only on  $\nu_f$  and  $\nu_t$ . The map  $d_{T_f, T_t}^0$  has rank  $\dim \text{Hom}_{\mathbf{S}}(T_f, T_t) - \dim \text{Hom}_{\Lambda}(T_f, T_t)$ . In addition,  $\text{Ext}_{\Lambda}^1(T_f, T_t) \cong \ker d_{T_f, T_t}^1 / \text{im } d_{T_f, T_t}^0$ . Using Crawley-Boevey's formula (4.2) and using the axiom (T1) of torsion pairs, we easily compute

$$\dim \ker d_{M, N}^1 = \dim \text{Ext}_{\Lambda}^1(T_f, T_t) + \text{rk } d_{T_f, T_t}^0 = \dim \text{Hom}_{\mathbf{S}}(T_f, T_t) - (\nu_f, \nu_t).$$

Remarkably, this dimension depends only on  $\nu_f$  and  $\nu_t$ , and not on the datum of the arrows  $(T_{t,a})$  and  $(T_{f,a})$ .

Let  $E$  be the set of all exact sequences  $0 \rightarrow T_t \xrightarrow{f} T \xrightarrow{g} T_f \rightarrow 0$  of  $I$ -graded vector spaces. This is a homogeneous space for the group  $G(\nu)$  and the stabilizer of a point  $(f, g)$  is  $\{\text{id} + fhg \mid h \in \text{Hom}_{\mathbf{S}}(T_f, T_t)\}$ . It is thus a smooth connected variety of dimension  $\dim G(\nu) - \dim \text{Hom}_{\mathbf{S}}(T_f, T_t)$ .

The fiber of  $p$  over a point  $((T_{t,a}), (T_{f,a})) \in \Lambda^{\mathcal{T}}(\nu_t) \times \Lambda^{\mathcal{F}}(\nu_f)$  consists of the datum of  $(f, g) \in E$  and of  $(T_a) \in \Lambda(\nu)$ , with a compatibility condition between the two. The datum of  $(f, g)$  corresponds to a trivial fiber bundle over  $\Lambda^{\mathcal{T}}(\nu_t) \times \Lambda^{\mathcal{F}}(\nu_f)$  with fiber  $E$ . Let us now examine how  $(T_a)$  can be chosen when  $((T_{t,a}), (T_{f,a}))$  and  $(f, g)$  are given.

Once chosen an  $I$ -graded complementary subspace of  $\ker g$  in the vector space  $T$ , the set of possible choices for  $(T_a)$  is isomorphic to  $\ker d_{T_f, T_t}^1$ ; moreover, the isomorphism depends smoothly on  $(f, g)$ . The linear map  $d_{T_f, T_t}^1$  depends smoothly on  $((T_{t,a}), (T_{f,a}))$  and has constant rank, as we have seen above, so its kernel depends smoothly on  $((T_{t,a}), (T_{f,a}))$ . In this fashion, we eventually see that the set of possible choices for  $(T_a)$  depends smoothly on  $((T_{t,a}), (T_{f,a}), f, g)$ . Choosing trivializations where needed, we conclude that  $p$  is a locally trivial fibration.

Finally, the dimension of the fibers of  $p$  is the sum of two contributions, namely  $\dim E$  and  $\dim \ker d_{T_f, T_t}^1$ . We find  $\dim G(\nu) - (\nu_f, \nu_t)$ , as announced.  $\square$

Let  $(Z_t, Z_f) \in \text{Irr } \Lambda^{\mathcal{T}}(\nu_t) \times \text{Irr } \Lambda^{\mathcal{F}}(\nu_f)$ . In view of Lemma 4.3,  $p^{-1}(Z_t \times Z_f)$  is an irreducible component of  $\Theta(\nu_t, \nu_f)$ . Then  $Z = q(p^{-1}(Z_t \times Z_f))$  is an irreducible subset of  $\Lambda(\nu)$  which, by e.g. I, §8, Theorem 3 in [44], has dimension

$$\dim(Z_t \times Z_f) + (\dim G(\nu) - (\nu_t, \nu_f)) - \dim(G(\nu_t) \times G(\nu_f)),$$

Equation (4.5) shows that this dimension is equal to that of  $\Lambda(\nu)$ , so  $\overline{Z} \in \text{Irr } \Lambda(\nu)$ .

This construction defines a map  $(\overline{Z}_t, \overline{Z}_f) \mapsto \overline{Z}$  from  $\mathfrak{T}(\nu_t) \times \mathfrak{F}(\nu_f)$  to  $\mathfrak{B}(\nu)$ . Gluing these maps for all possible  $(\nu_t, \nu_f)$ , we eventually get a map  $\Xi : \mathfrak{T} \times \mathfrak{F} \rightarrow \mathfrak{B}$ .

**Theorem 4.4** *The map  $\Xi : \mathfrak{T} \times \mathfrak{F} \rightarrow \mathfrak{B}$  is bijective.*

*Proof.* Given  $\nu_t$  and  $\nu_f$ ,  $\Xi$  is a bijection from  $\mathfrak{T}(\nu_t) \times \mathfrak{F}(\nu_f)$  onto the set of irreducible components of  $\overline{q(\Theta(\nu_t, \nu_f))}$ .

Now we take  $\nu \in \text{NI}$ . We consider the diagrams (4.6) for all  $\nu_t$  and  $\nu_f$  such that  $\nu_t + \nu_f = \nu$ . In this fashion, we split  $\Lambda(\nu)$  according to the dimension-vector of the torsion submodule:

$$\Lambda(\nu) = \bigsqcup_{\nu_t + \nu_f = \nu} q(\theta(\nu_t, \nu_f)).$$

Each piece of this partition is constructible, therefore an irreducible component of  $\Lambda(\nu)$  is contained in one and only one closure  $\overline{q(\Theta(\nu_t, \nu_f))}$ .

Taking the union, we see that  $\Xi$  defines a bijection from  $\bigsqcup_{\nu_t + \nu_f = \nu} (\mathfrak{T}(\nu_t) \times \mathfrak{F}(\nu_f))$  onto  $\mathfrak{B}(\nu)$ .  $\square$

*Remark 4.5.* The construction of  $\Xi$  implies that if  $T$  is a general point of  $\overline{Z}$ , then the torsion submodule  $X$  of  $T$  has dimension-vector  $\nu_t$  and the point  $(X, T/X)$  is general in  $\overline{Z}_t \times \overline{Z}_f$ . To see this, take a  $G(\nu_t)$ -invariant dense open subset  $U_t \subseteq Z_t$  and a  $G(\nu_f)$ -invariant dense open subset of  $U_f \subseteq Z_f$ . Then  $p^{-1}(U_t \times U_f)$  is dense in  $p^{-1}(Z_t \times Z_f)$ . The subset  $q(p^{-1}(U_t \times U_f))$  is thus dense in the irreducible set  $\overline{Z}$ , and is constructible by Chevalley's theorem, so it contains a dense open subset  $U$  of  $\overline{Z}$ . By construction, if  $T$  belongs to  $U$ , then  $\underline{\dim} X = \nu_t$  and  $(X, T/X)$  belongs to the prescribed open subset  $U_t \times U_f$ , as desired.

Suppose now that we are given two torsion pairs  $(\mathcal{T}', \mathcal{F}')$  and  $(\mathcal{T}'', \mathcal{F}'')$  in  $\Lambda\text{-mod}$  that both satisfy the openness condition (O). They give rise to subsets  $\mathfrak{T}', \mathfrak{F}', \mathfrak{T}''$  and  $\mathfrak{F}''$  of  $\mathfrak{B}$  and to bijections  $\Xi' : \mathfrak{T}' \times \mathfrak{F}' \rightarrow \mathfrak{B}$  and  $\Xi'' : \mathfrak{T}'' \times \mathfrak{F}'' \rightarrow \mathfrak{B}$ .

**Proposition 4.6** *Assume that  $(\mathcal{T}', \mathcal{F}') \preceq (\mathcal{T}'', \mathcal{F}'')$ . Then the map  $\Xi'$  restricts to a bijection  $\mathfrak{T}' \times (\mathfrak{F}' \cap \mathfrak{T}'') \rightarrow \mathfrak{T}''$ , the map  $\Xi''$  restricts to a bijection  $(\mathfrak{F}' \cap \mathfrak{T}'') \times \mathfrak{F}'' \rightarrow \mathfrak{F}'$ , and we have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{T}' \times (\mathfrak{F}' \cap \mathfrak{T}'') \times \mathfrak{F}'' & \xrightarrow{\Xi' \times \text{id}} & \mathfrak{T}'' \times \mathfrak{F}'' \\ \text{id} \times \Xi'' \downarrow & & \downarrow \Xi'' \\ \mathfrak{T}' \times \mathfrak{F}' & \xrightarrow{\Xi'} & \mathfrak{B}. \end{array}$$

*Proof.* Let  $(Z_1, Z_2) \in \mathfrak{T}' \times \mathfrak{F}'$  and set  $Z = \Xi'(Z_1, Z_2)$ . Let  $T$  be a general point of  $Z$  and let  $X$  be the torsion submodule of  $T$  with respect to  $(\mathcal{T}', \mathcal{F}')$ . Then  $X \in \mathcal{T}''$  and the point  $(X, T/X)$  is general in  $Z_1 \times Z_2$ . Since a torsion class is stable under taking quotients and extensions,  $T$  belongs to  $\mathcal{T}''$  if and only if  $T/X$  does. This means that  $Z$  belongs to  $\mathfrak{T}''$  if and only if  $Z_2$  does. Thus  $\Xi'$  restricts to a bijection  $\mathfrak{T}' \times (\mathfrak{F}' \cap \mathfrak{T}'') \rightarrow \mathfrak{T}''$ , as announced.

One shows that  $\Xi''$  restricts to a bijection  $(\mathfrak{F}' \cap \mathfrak{T}'') \times \mathfrak{F}'' \rightarrow \mathfrak{F}'$  in a similar fashion.

Now let  $(Z_1, Z_2, Z_3) \in \mathfrak{T}' \times (\mathfrak{F}' \cap \mathfrak{T}'') \times \mathfrak{F}''$ . Set  $Z_4 = \Xi'(Z_1, Z_2)$  and  $Z = \Xi''(Z_4, Z_3)$ . Let  $T$  be a general point of  $Z$  and let  $X'$  and  $X''$  be the torsion submodules of  $T$  with respect to  $(\mathcal{T}', \mathcal{F}')$  and  $(\mathcal{T}'', \mathcal{F}'')$ , respectively. Then the point  $(X'', T/X'')$  is general in  $Z_4 \times Z_3$ . Since  $X'$  is the torsion submodule of  $X''$  with respect to  $(\mathcal{T}', \mathcal{F}')$ , the point  $(X', X''/X', T/X'')$  is general in  $Z_1 \times Z_2 \times Z_3$ .

A similar reasoning shows that  $(X', X''/X', T/X'')$  is also general in  $\tilde{Z}_1 \times \tilde{Z}_2 \times \tilde{Z}_3$ , where  $(\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3) = (\Xi' \circ (\text{id} \times \Xi''))^{-1}(Z)$ . Therefore a point can be general in  $Z_1 \times Z_2 \times Z_3$  and in  $\tilde{Z}_1 \times \tilde{Z}_2 \times \tilde{Z}_3$  at the same time. Then necessarily  $(Z_1, Z_2, Z_3) = (\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3)$ , which establishes the commutativity.  $\square$

The torsion pairs  $(\mathcal{I}_\theta, \overline{\mathcal{P}}_\theta)$  and  $(\overline{\mathcal{T}}_\theta, \mathcal{P}_\theta)$  satisfy the assumption (O), thanks to Proposition 4.2 (iii). Applying Proposition 4.6 to them, we get a bijection

$$\Xi_\theta : \mathfrak{I}_\theta \times \mathfrak{R}_\theta \times \mathfrak{P}_\theta \rightarrow \mathfrak{B},$$

where  $\mathfrak{I}_\theta$ ,  $\mathfrak{R}_\theta$  and  $\mathfrak{P}_\theta$  are the subsets of  $\mathfrak{B}$ , consisting of components whose general points belong to  $\mathcal{I}_\theta$ ,  $\mathcal{R}_\theta$  and  $\mathcal{P}_\theta$ , respectively.

We also note that Proposition 4.6 can be generalized in an obvious fashion to any finite nested sequence

$$(\mathcal{T}_0, \mathcal{F}_0) \preceq \cdots \preceq (\mathcal{T}_\ell, \mathcal{F}_\ell)$$

of torsion pairs that satisfy (O).

## 5 Tilting theory on preprojective algebras

### 5.1 Reflection functors

Let  $I_i = \Lambda(1 - e_i)\Lambda$ ; in other words, let  $I_i$  be the annihilator of the simple  $\Lambda$ -module  $S_i$ . Amplifying their previous work [28], Iyama and Reiten, in a joint paper [13] with Buan and Scott, show that  $I_i$  is a tilting  $\Lambda$ -module of projective dimension at most one and that  $\text{End}_\Lambda(I_i) \cong \Lambda$  (at least, when no connected component of  $(I, E)$  is of Dynkin type).

This certainly invites to look at the endofunctors  $\Sigma_i = \text{Hom}_\Lambda(I_i, ?)$  and  $\Sigma_i^* = I_i \otimes_\Lambda ?$  of the category  $\Lambda\text{-mod}$ . It turns out that these functors can be described in a very explicit fashion.

Recall that we locally depict a  $\Lambda$ -module  $M$  around the vertex  $i$  by the diagram (4.1).

**Proposition 5.1** (i) *The module  $\Sigma_i M$  is obtained by replacing (4.1) with*

$$\widetilde{M}_i \xrightarrow{\overline{M}_{\text{out}(i)} M_{\text{in}(i)}} \ker M_{\text{in}(i)} \hookrightarrow \widetilde{M}_i,$$

where the map  $\overline{M}_{\text{out}(i)} : M_i \rightarrow \ker M_{\text{in}(i)}$  is induced by  $M_{\text{out}(i)}$ .

(ii) *The module  $\Sigma_i^* M$  is obtained by replacing (4.1) with*

$$\widetilde{M}_i \twoheadrightarrow \text{coker } M_{\text{out}(i)} \xrightarrow{M_{\text{out}(i)} \overline{M}_{\text{in}(i)}} \widetilde{M}_i,$$

where the map  $\overline{M}_{\text{in}(i)} : \text{coker } M_{\text{out}(i)} \rightarrow M_i$  is induced by  $M_{\text{in}(i)}$ .

*Proof.* Applying the functor  $S_i \otimes_\Lambda ?$  to the resolution (4.3) and changing the right arrow by a sign, one finds the following exact sequence of right  $\Lambda$ -modules.

$$K\rho_i \otimes_{\mathbf{S}} \Lambda \xrightarrow{\partial_1} \bigoplus_{\substack{a \in H \\ s(a)=i}} Ka^* \otimes_{\mathbf{S}} \Lambda \xrightarrow{\partial_0} e_i I_i \rightarrow 0, \quad (5.1)$$

where

$$\partial_1(\rho_i \otimes 1) = \sum_{\substack{a \in H \\ s(a)=i}} \varepsilon(a) a^* \otimes a \quad \text{and} \quad \partial_0(a^* \otimes 1) = a^*.$$

The sequence obtained by applying  $? \otimes_\Lambda M$  to (5.1) can be identified with

$$M_i \xrightarrow{M_{\text{out}(i)}} \widetilde{M}_i \rightarrow e_i I_i \otimes_\Lambda M \rightarrow 0.$$

Using the decomposition  $I_i = (1 - e_i)\Lambda \oplus e_i I_i$ , one can subsequently identify  $\Sigma_i^* M = I_i \otimes_\Lambda M$  with the vector space described in Statement (ii).

Let us check the equality  $(\Sigma_i^* M)_{\text{out}(i)} = M_{\text{out}(i)} \overline{M}_{\text{in}(i)}$ . Let  $x \in \text{coker } M_{\text{out}(i)}$ . It can be represented by an element  $(x_b) \in \widetilde{M}_i$ , which, in the identification

$$\widetilde{M}_i \cong \left( \bigoplus_{\substack{b \in H \\ s(b)=i}} K b^* \otimes_{\mathbf{S}} \Lambda \right) \otimes_\Lambda M, \quad \text{corresponds to} \quad \sum_{\substack{b \in H \\ s(b)=i}} b^* \otimes x_b.$$

In the  $\Lambda$ -module  $\Sigma_i^* M$ , an arrow  $a$  that starts at  $i$  maps  $x$  to

$$\sum_{\substack{b \in H \\ s(b)=i}} (ab^*) \otimes x_b = \sum_{\substack{b \in H \\ s(b)=i}} M_a M_{b^*} x_b,$$

where the left-hand side lives in  $(1 - e_i)\Lambda \otimes_\Lambda M$ . Therefore

$$(\Sigma_i^* M)_{\text{out}(i)}(x) = \left( \sum_{\substack{b \in H \\ s(b)=i}} \varepsilon(a) M_a M_{b^*} x_b \right)_{\substack{a \in H \\ s(a)=i}} = M_{\text{out}(i)} \overline{M}_{\text{in}(i)}(x).$$

One checks similarly that  $(\Sigma_i^* M)_{\text{in}(i)}$  is the canonical map  $\widetilde{M}_i \rightarrow \text{coker } M_{\text{out}(i)}$ , which concludes the proof of (ii). The proof of (i) is similar, with two differences: one starts with the exact sequence obtained by applying  $?\otimes_\Lambda S_i$  to the resolution (4.3), and one has to change the position of the signs  $\varepsilon(a)$  (Remark 2.4 in [5] explains that this change is without consequences).  $\square$

These mutually adjoint functors  $\Sigma_i$  and  $\Sigma_i^*$  are called reflection functors. The concrete description afforded by the proposition yields several important properties that they enjoy:

- Adjunction morphisms can be inserted in functorial short exact sequences

$$0 \rightarrow \text{soc}_i \rightarrow \text{id} \rightarrow \Sigma_i \Sigma_i^* \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \Sigma_i^* \Sigma_i \rightarrow \text{id} \rightarrow \text{hd}_i \rightarrow 0, \quad (5.2)$$

where the  $i$ -socle  $\text{soc}_i M$  and the  $i$ -head  $\text{hd}_i M$  of a  $\Lambda$ -module  $M$  are defined to be the kernel of  $M_{\text{out}(i)}$  and the cokernel of  $M_{\text{in}(i)}$ , respectively (see Proposition 2.5 in [5]).

- They are exchanged by the  $*$ -duality; in other words,  $\Sigma_i^* T^* \cong (\Sigma_i T)^*$  for all finite dimensional  $\Lambda$ -module  $T$ .

- They induce Kashiwara and Saito's crystal reflections  $S_i$  and  $S_i^*$  on  $\mathfrak{B}$  (see Section 5.5).
- The operation of restricting a representation of  $\Lambda$  to a representation of the single quiver  $Q$  intertwines the functors  $\Sigma_i$  and  $\Sigma_i^*$  with the traditional Bernstein-Gelfand-Ponomarev reflection functors (see Proposition 7.1 in [5]).

*Remark 5.2.* These functors  $\Sigma_i$  and  $\Sigma_i^*$  were introduced by Iyama and Reiten in [29] by means of the ideals  $I_i$ , and also, independently, by the first two authors in [5] by the explicit description of Proposition 5.1. The link between the two constructions were suggested to us by Amiot.

## 5.2 The tilting ideals $I_w$

Reflection functors satisfy the braid relations, so it is natural to study products of reflection functors computed according to reduced decompositions of elements in  $W$ . We now look for an analog of the exact sequences (5.2) for such a product.

To simplify the presentation, we consider in this section the case where no connected component of the diagram  $(I, E)$  is of Dynkin type. This allows us to rely on the following result, due to Buan, Iyama, Reiten and Scott (Section II.1 in [13]; see also Lemma 2.5 in [52] for the second statement of assertion (iv)). We will however argue in Section 5.6 that almost all the results presented here hold true in general.

- Theorem 5.3** (i) *If  $s_{i_1} \cdots s_{i_\ell}$  is a reduced decomposition, then the multiplication in  $\Lambda$  gives rise to an isomorphism of bimodules  $I_{i_1} \otimes_\Lambda \cdots \otimes_\Lambda I_{i_\ell} \rightarrow I_{i_1} \cdots I_{i_\ell}$ .*
- (ii) *Under the same assumption, the product  $I_{i_1} \cdots I_{i_\ell}$  depends only on  $w = s_{i_1} \cdots s_{i_\ell}$ ; we can thus denote it by  $I_w$ . It has finite codimension in  $\Lambda$ .*
- (iii) *Each  $I_w$  is a tilting  $\Lambda$ -bimodule of projective dimension at most 1 and  $\text{End}_\Lambda(I_w) \cong \Lambda$ .*
- (iv) *If  $\ell(ws_i) > \ell(w)$ , then  $\text{Tor}_1^\Lambda(I_w, S_i) = 0$ . If  $\ell(s_i w) > \ell(w)$ , then  $\text{Ext}_\Lambda^1(I_w, S_i) = 0$ .*

In view of Theorem 5.3 (iii), it is natural to apply Brenner and Butler's theorem. For that, we define categories

$$\begin{aligned} \mathcal{T}^w &= \mathcal{T}(I_w) = \{T \mid \text{Ext}_\Lambda^1(I_w, T) = 0\}, \\ \mathcal{F}^w &= \mathcal{F}(I_w) = \{T \mid \text{Hom}_\Lambda(I_w, T) = 0\}, \\ \mathcal{T}_w &= \mathcal{X}(I_w) = \{T \mid I_w \otimes_\Lambda T = 0\}, \\ \mathcal{F}_w &= \mathcal{Y}(I_w) = \{T \mid \text{Tor}_\Lambda^1(I_w, T) = 0\}. \end{aligned}$$



**Theorem 5.4** (i) The pair  $(\mathcal{T}^w, \mathcal{F}^w)$  is a torsion pair in  $\Lambda\text{-mod}$ . For each  $\Lambda$ -module  $T$ , the evaluation map  $I_w \otimes_{\Lambda} \text{Hom}_{\Lambda}(I_w, T) \rightarrow T$  is injective and its image is the torsion submodule of  $T$  with respect to  $(\mathcal{T}^w, \mathcal{F}^w)$ .

(ii) The pair  $(\mathcal{T}_w, \mathcal{F}_w)$  is a torsion pair in  $\Lambda\text{-mod}$ . For each  $\Lambda$ -module  $T$ , the coevaluation map  $T \rightarrow \text{Hom}_{\Lambda}(I_w, I_w \otimes_{\Lambda} T)$  is surjective and its kernel is the torsion submodule of  $T$  with respect to  $(\mathcal{T}_w, \mathcal{F}_w)$ .

(iii) There are mutually inverse equivalences

$$\mathcal{F}_w \xrightleftharpoons[\text{Hom}_{\Lambda}(I_w, ?)]{I_w \otimes_{\Lambda} ?} \mathcal{T}^w .$$

*Proof.* See [3], in particular the lemma in Section 1.6, the corollary in Section 1.9, and the theorem and its corollary in Section 2.1.  $\square$

Given  $w \in W$  and  $T \in \Lambda\text{-mod}$ , we will denote by  $T^w$  and  $T_w$  the torsion submodules of  $T$  with respect to  $(\mathcal{T}^w, \mathcal{F}^w)$  and  $(\mathcal{T}_w, \mathcal{F}_w)$ , respectively.

If  $u, v \in W$  are such that  $\ell(u) + \ell(v) = \ell(uv)$ , then  $I_{uv} \cong I_u \otimes_{\Lambda} I_v$ . It immediately follows that

$$\mathcal{F}^{uv} \supseteq \mathcal{F}^u \quad \text{and} \quad \mathcal{T}_v \subseteq \mathcal{T}_{uv},$$

which can be written

$$(\mathcal{T}^{uv}, \mathcal{F}^{uv}) \preceq (\mathcal{T}^u, \mathcal{F}^u) \quad \text{and} \quad (\mathcal{T}_v, \mathcal{F}_v) \preceq (\mathcal{T}_{uv}, \mathcal{F}_{uv}). \quad (5.3)$$

*Remarks 5.5.* (i) The reader may here object that we apply a theorem valid for finite dimensional algebras to an infinite dimensional framework. In fact, there is no difficulty. The non-obvious point is to show that the functors  $\text{Hom}_{\Lambda}(I_w, ?)$  and  $I_w \otimes_{\Lambda} ?$  preserve the category of finite dimensional  $\Lambda$ -modules. In the case where  $w$  is a simple reflection, this follows from Proposition 5.1. The general case follows by composition.

(ii) By Theorem 5.4 (iii), any module  $T \in \mathcal{T}^w$  is isomorphic to a module of the form  $I_w \otimes_{\Lambda} X$ . Conversely, one easily checks that if  $T$  is of the form  $I_w \otimes_{\Lambda} X$ , then the evaluation map  $I_w \otimes_{\Lambda} \text{Hom}_{\Lambda}(I_w, T) \rightarrow T$  is surjective, so  $T$  belongs to  $\mathcal{T}^w$  by Theorem 5.4 (i). Therefore  $\mathcal{T}^w$  is the essential image of the functor  $I_w \otimes_{\Lambda} ?$ . Likewise, one shows that  $\mathcal{F}_w$  is the essential image of  $\text{Hom}_{\Lambda}(I_w, ?)$ .

*Examples 5.6.* (i) The case where  $w$  is a simple reflection has been dealt with in the previous section: the functorial short exact sequences (5.2) imply that the torsion submodule of  $T$  with respect to  $(\mathcal{T}_{s_i}, \mathcal{F}_{s_i})$  is the  $i$ -socle of  $T$  and that the torsion-free quotient of  $T$  with respect to  $(\mathcal{T}^{s_i}, \mathcal{F}^{s_i})$  is the  $i$ -head of  $T$ . Therefore

$$\begin{aligned}\mathcal{T}^{s_i} &= \mathcal{T}(I_i) = \{T \mid \text{hd}_i T = 0\}, \\ \mathcal{F}^{s_i} &= \mathcal{F}(I_i) = \text{add } S_i, \\ \mathcal{T}_{s_i} &= \mathcal{X}(I_i) = \text{add } S_i, \\ \mathcal{F}_{s_i} &= \mathcal{Y}(I_i) = \{T \mid \text{soc}_i T = 0\}.\end{aligned}$$

These equalities were also obtained by Sekiya and Yamaura (Lemmas 2.15 and 2.16 in [52]).

- (ii) Let us generalize the first example. Let  $J \subseteq I$  and let  $Q_J$  be the full subquiver  $Q$  with set of vertices  $J$ . The preprojective algebra  $\Lambda_J$  is a quotient of  $\Lambda$ . The kernel of the natural morphism  $\Lambda \rightarrow \Lambda_J$  is the ideal  $I_J = \Lambda(1 - \sum_{j \in J} e_j)\Lambda$ . The pull-back functor allows to identify  $\Lambda_J\text{-mod}$  with the full subcategory

$$\{M \in \Lambda\text{-mod} \mid M_j \neq 0 \Rightarrow j \in J\}$$

of  $\Lambda\text{-mod}$ . Moreover, a  $\Lambda$ -module  $T$  belongs to  $\Lambda_J\text{-mod}$  if and only if  $\text{Hom}_\Lambda(I_J, T) = 0$ . In fact, if the latter equality holds, then  $I_J T = 0$ , hence  $T$  is a  $\Lambda/I_J$ -module. Conversely, if  $T \in \Lambda_J\text{-mod}$ , then

$$\text{Ext}_\Lambda^1(\Lambda_J, T) \cong \text{Ext}_{\Lambda_J}^1(\Lambda_J, T) = 0,$$

for  $\Lambda_J\text{-mod}$  is closed under extensions; since the first arrow in the short exact sequence

$$\text{Hom}_\Lambda(\Lambda_J, T) \hookrightarrow \text{Hom}_\Lambda(\Lambda, T) \rightarrow \text{Hom}_\Lambda(I_J, T) \rightarrow \text{Ext}_\Lambda^1(\Lambda_J, T) = 0$$

is an isomorphism, we get  $\text{Hom}_\Lambda(I_J, T) = 0$ .

Assume now that  $Q_J$  is of Dynkin type. Then Theorem II.3.5 in [13] says that  $I_J$  is the ideal  $I_{w_J}$ , where  $w_J$  is the longest element in the parabolic subgroup  $W_J = \langle s_j \mid j \in J \rangle$  of  $W$ . We conclude that  $\mathcal{F}^{w_J} = \Lambda_J\text{-mod}$ . Further, by duality, we also have  $\mathcal{T}_{w_J} = \Lambda_J\text{-mod}$ .

Thus the torsion-free part of a module  $T$  with respect to the torsion pair  $(\mathcal{T}^{w_J}, \mathcal{F}^{w_J})$  is the largest quotient of  $T$  that belongs to  $\Lambda_J\text{-mod}$ , and the torsion part of  $T$  with respect to the torsion pair  $(\mathcal{T}_{w_J}, \mathcal{F}_{w_J})$  is the largest submodule of  $T$  that belongs to  $\Lambda_J\text{-mod}$ .

- (iii) We have already mentioned that the reflection functors  $\Sigma_i$  and  $\Sigma_i^*$  are exchanged by the  $*$ -duality. By composition, we obtain  $(I_w \otimes_\Lambda T)^* = \text{Hom}_\Lambda(I_{w^{-1}}, T^*)$ . This readily implies

$$\mathcal{F}^w = (\mathcal{T}_{w^{-1}})^* \quad \text{and} \quad \mathcal{T}^w = (\mathcal{F}_{w^{-1}})^*.$$

- (iv) We will explain in Examples 5.14 and 5.15 that  $\mathcal{F}^w$  is Buan, Iyama, Reiten and Scott's category  $\text{Sub}(\Lambda/I_w)$  [13] and that  $\mathcal{T}_w$  is Geiß, Leclerc and Schröer's category  $\mathcal{C}_w$  [24].

We conclude this section by showing that the equivalence of categories described by Theorem 5.4 (iii) can be broken into pieces according to any reduced decomposition of  $w$ .

**Proposition 5.7** *Let  $(u, v, w) \in W^3$  be such that  $\ell(uvw) = \ell(u) + \ell(v) + \ell(w)$ . Then one has mutually inverse equivalences*

$$\mathcal{F}_{uv} \cap \mathcal{T}^w \xrightleftharpoons[\text{Hom}_\Lambda(I_v, ?)]{I_v \otimes_\Lambda ?} \mathcal{F}_u \cap \mathcal{T}^{vw}.$$

*Proof.* Certainly,  $I_v \otimes_\Lambda ?$  maps  $\mathcal{T}^w$ , the essential image of  $I_w \otimes_\Lambda ?$ , to  $\mathcal{T}^{vw}$ , the essential image of  $I_{vw} \otimes_\Lambda ?$ . On the other side, any module  $T \in \mathcal{F}_{uv}$  belongs to the essential image of

$$\text{Hom}_\Lambda(I_{uv}, ?) = \text{Hom}_\Lambda(I_v, \text{Hom}_\Lambda(I_u, ?)),$$

hence is isomorphic to  $\text{Hom}_\Lambda(I_v, X)$ , with  $X \in \mathcal{F}_u$ . By Theorem 5.4 (i),  $I_v \otimes_\Lambda T$  is a submodule of  $X$ , hence belongs to  $\mathcal{F}_u$ . To sum up,  $I_v \otimes_\Lambda ?$  maps  $\mathcal{T}^w$  to  $\mathcal{T}^{vw}$  and maps  $\mathcal{F}_{uv}$  to  $\mathcal{F}_u$ , hence maps  $\mathcal{F}_{uv} \cap \mathcal{T}^w$  to  $\mathcal{F}_u \cap \mathcal{T}^{vw}$ .

One shows in a dual fashion that  $\text{Hom}_\Lambda(I_v, ?)$  maps  $\mathcal{F}_u \cap \mathcal{T}^{vw}$  to  $\mathcal{F}_{uv} \cap \mathcal{T}^w$ .  $\square$

Under the assumptions of the proposition, one thus has a chain of equivalences of categories

$$\mathcal{F}_{uvw} \xrightleftharpoons[\text{Hom}_\Lambda(I_w, ?)]{I_w \otimes_\Lambda ?} \mathcal{F}_{uv} \cap \mathcal{T}^w \xrightleftharpoons[\text{Hom}_\Lambda(I_v, ?)]{I_v \otimes_\Lambda ?} \mathcal{F}_u \cap \mathcal{T}^{vw} \xrightleftharpoons[\text{Hom}_\Lambda(I_u, ?)]{I_u \otimes_\Lambda ?} \mathcal{T}^{uvw}.$$

**Corollary 5.8** *Let  $w \in W$  and let  $T \in \mathcal{F}_w$ . Then*

$$\underline{\dim} I_w \otimes_\Lambda T = w(\underline{\dim} T).$$

*Proof.* The particular case where  $w$  is a simple reflection is Lemma 2.5 in [1]; it can also be proved in a more elementary way with the help of Proposition 5.1. The general case is obtained by writing  $w$  as a product of  $\ell(w)$  simple reflections.  $\square$

*Remark 5.9.* Theorem 5.4 can also be proved in a more elementary fashion, without appealing to tilting theory. Instead of invoking Brenner and Butler's theorem, one can indeed apply  $\ell(w)$

times the functorial short exact sequences (5.2), twisting at each step by an adequate product of reflection functors. A glimpse of this method will appear during the proof of Theorem 5.11 below. The key fact is that  $\text{Tor}_1^\Lambda(I_w, S_i) = 0$  when  $\ell(ws_i) > \ell(w)$ . In the finite type case, one can show this vanishing by controlling dimension-vectors. This approach was in fact our original proof to nearly all the results presented in this Section 5, before we became aware that they were closely related to the works [1], [13], [24] and [29].

### 5.3 Layers and stratification

In Section 10 of [24], Geiß, Leclerc and Schröer construct a filtration of the objects in  $\mathcal{T}_w$ . In Section 2 of [1], Amiot, Iyama, Reiten and Todorov construct a filtration of the finite dimensional module  $\Lambda/I_w$  by layers. Our aim in this section is to show that these two constructions are  $*$ -dual to each other and are defined by the torsion pairs associated to the tilting ideals  $I_w$ . These results will help us later to identify the categories  $\mathcal{T}^w$ ,  $\mathcal{F}^w$ ,  $\mathcal{T}_w$  and  $\mathcal{F}_w$  with the categories  $\mathcal{I}_\theta$  and  $\mathcal{P}_\theta$  from Section 3.1.

We begin by introducing the layers. The next lemma is identical to Corollary 9.3 in [24], to Theorem 2.6 in [1], and to Proposition 5.2 in [52]. We cannot resist offering a fourth proof of this basic result.

**Lemma 5.10** *Let  $(w, i) \in W \times I$  be such that  $\ell(ws_i) > \ell(w)$ . Then the simple module  $S_i$  belongs to  $\mathcal{F}_w$ . Moreover, the module  $I_w \otimes_\Lambda S_i$  belongs to  $\mathcal{F}^{ws_i}$  and has dimension-vector  $w\alpha_i$ .*

*Proof.* We proceed by induction on the length of  $w$ . The case  $w = 1$  is obvious. Assume  $\ell(w) > 0$  and write  $w = s_j v$  with  $\ell(v) < \ell(w)$ . Applying  $\text{Hom}_\Lambda(I_v, ?)$  to the short exact sequence

$$0 \rightarrow \text{soc}_j(I_v \otimes_\Lambda S_i) \rightarrow I_v \otimes_\Lambda S_i \rightarrow \Sigma_j \Sigma_j^*(I_v \otimes_\Lambda S_i) \rightarrow 0$$

leads to

$$0 \rightarrow \text{Hom}_\Lambda(I_v, S_j)^{\oplus n} \rightarrow \text{Hom}_\Lambda(I_v, I_v \otimes_\Lambda S_i) \rightarrow \text{Hom}_\Lambda(I_w, I_w \otimes_\Lambda S_i) \rightarrow 0$$

with  $n = \dim \text{soc}_j(I_v \otimes_\Lambda S_i)$ , by Theorem 5.3 (iv). Applied to  $(v, i)$ , the inductive hypothesis says that the middle term is isomorphic to  $S_i$ , hence is simple. Applied to  $(v^{-1}, j)$ , the inductive hypothesis, together with Corollary 5.8, implies that the left term has dimension-vector  $n v^{-1}(\alpha_j)$ . Moreover, the assumption  $\ell(ws_i) > \ell(w)$  means that  $v^{-1}(\alpha_j) \neq \alpha_i$ . We conclude that necessarily  $n = 0$  and that the right term is isomorphic to  $S_i$ . Therefore  $S_i$  belongs to the essential image of  $\text{Hom}_\Lambda(I_w, ?)$ , that is, to  $\mathcal{F}_w$ . We also see that

$$\text{Hom}_\Lambda(I_{ws_i}, I_w \otimes_\Lambda S_i) \cong \Sigma_i^* S_i = 0,$$

which means that  $I_w \otimes_\Lambda S_i$  belongs to  $\mathcal{F}^{ws_i}$ . The last claim follows from Corollary 5.8.  $\square$

The following result is in essence due to Amiot, Iyama, Reiten and Todorov [1] and to Geiß, Leclerc and Schröer [24].

**Theorem 5.11** (i) *Let  $(w, i) \in W \times I$  be such that  $\ell(ws_i) > \ell(w)$  and let  $T \in \Lambda\text{-mod}$ . Then  $T^w/T^{ws_i}$  is the largest quotient of  $T^w$  isomorphic to a direct sum of copies of the module  $I_w \otimes_\Lambda S_i$ .*

(ii) *Let  $w = s_{i_1} \cdots s_{i_\ell}$  be a reduced decomposition. Then a module  $T \in \Lambda\text{-mod}$  belongs to  $\mathcal{F}^w$  if and only if it has a filtration whose subquotients are modules of the form  $I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_\Lambda S_{i_k}$  with  $1 \leq k \leq \ell$ .*

(iii) *Let  $w = s_{i_1} \cdots s_{i_\ell}$  be a reduced decomposition. Then a module  $T \in \Lambda\text{-mod}$  belongs to  $\mathcal{T}^w$  if and only if there is no epimorphism  $T \twoheadrightarrow I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_\Lambda S_{i_k}$  with  $1 \leq k \leq \ell$ .*

*Proof.* Let  $w, i$  and  $T$  as in (i) and set  $X = \text{Hom}_\Lambda(I_w, T)$ . Applying  $I_w \otimes_\Lambda ?$  to the short exact sequence

$$0 \rightarrow \Sigma_i^* \Sigma_i X \rightarrow X \rightarrow \text{hd}_i X \rightarrow 0$$

and using  $\text{Tor}_1^\Lambda(I_w, S_i) = 0$  (Theorem 5.3 (iv)), we get

$$0 \rightarrow I_{ws_i} \otimes_\Lambda \text{Hom}_\Lambda(I_{ws_i}, T) \rightarrow I_w \otimes_\Lambda \text{Hom}_\Lambda(I_w, T) \rightarrow I_w \otimes_\Lambda S_i^{\oplus n} \rightarrow 0,$$

where  $n = \dim \text{hd}_i X$ . Therefore  $T^w/T^{ws_i}$  has the desired form.

To finish the proof of (i), it remains to show the maximality. Before that, we look at Statements (ii) and (iii). For  $1 \leq k \leq \ell$ , we set  $L_k = I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_\Lambda S_{i_k}$ .

Let  $T \in \Lambda\text{-mod}$ . What has already been showed from Statement (i) implies that  $T/T^w$  has a filtration whose subquotients are all isomorphic to some  $L_k$ . If  $T$  belongs to  $\mathcal{F}^w$ , then  $T = T/T^w$  has such a filtration: this shows the necessity of the condition proposed in Statement (ii). If  $T$  has no quotient isomorphic to a module  $L_k$ , then  $T = T^w$ , and therefore  $T$  belongs to  $\mathcal{T}^w$ : this shows the sufficiency of the condition proposed in Statement (iii).

By Lemma 5.10,  $L_k$  belongs to  $\mathcal{F}^{s_{i_1} \cdots s_{i_k}}$ , hence to  $\mathcal{F}^w$ . Any iterated extension of modules in the set  $\{L_k \mid 1 \leq k \leq \ell\}$  therefore belongs to  $\mathcal{F}^w$ , for  $\mathcal{F}^w$  is closed under extensions: this shows the sufficiency of the condition in Statement (ii). On the other hand, a module in  $\mathcal{T}^w$  cannot have a nonzero map to a module in  $\mathcal{F}^w$ , hence has no quotient isomorphic to a module  $L_k$ : this shows the necessity of the condition in Statement (iii).

Now let us go back to Statement (i), resuming the proof where we left it. We choose a reduced decomposition  $w = s_{i_1} \cdots s_{i_{\ell-1}}$  and set  $i_\ell = i$ . For  $1 \leq k \leq \ell$ , we set  $L_k = I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_\Lambda S_{i_k}$ .

Assume the existence of a short exact sequence  $0 \rightarrow Z \rightarrow T^w \rightarrow L_\ell^{\oplus n} \rightarrow 0$ , where  $Z$  does not belong to  $\mathcal{T}^w$ . Then  $Z$  has a quotient  $Z/Y$  isomorphic to some  $L_k$ , with  $k < \ell$ , and we have an extension  $0 \rightarrow L_k \rightarrow T^w/Y \rightarrow L_\ell^{\oplus n} \rightarrow 0$ . By Lemma 5.10,  $\underline{\dim} L_k$  and  $\underline{\dim} L_\ell$  are distinct real roots, hence are not proportional; therefore  $\underline{\dim} T^w/Y$  is not a multiple of  $\underline{\dim} L_\ell$ . Let us set  $U = T^w/Y$ . This module belongs to  $\mathcal{T}^w$ , because  $T^w$  does, so  $U = U^w$ ; it also belongs to  $\mathcal{F}^{ws_i}$ , because  $L_k$  and  $L_\ell$  do and because  $\mathcal{F}^{ws_i}$  is closed under extensions, so  $U^{ws_i} = 0$ . Therefore  $U = U^w/U^{ws_i}$ , which implies that  $U$  is a direct sum of copies of  $L_\ell$ . We thus reach a contradiction, and conclude that the assumption at the beginning of this paragraph is wrong.

Consider a short exact sequence  $0 \rightarrow Z \rightarrow T^w \rightarrow L_\ell^{\oplus n} \rightarrow 0$ . The preceding paragraph says that  $Z$  belongs to  $\mathcal{T}^w$ . Applying  $\text{Hom}_\Lambda(I_w, ?)$ , we deduce the short exact sequence  $0 \rightarrow \text{Hom}_\Lambda(I_w, Z) \rightarrow X \rightarrow S_i^{\oplus n} \rightarrow 0$ . Then, by definition of the  $i$ -head,  $\text{Hom}_\Lambda(I_w, Z)$  contains  $\Sigma_i^* \Sigma_i X$ ; moreover the quotient  $\text{Hom}_\Lambda(I_w, Z)/\Sigma_i^* \Sigma_i X$  is a direct sum of copies of  $S_i$ . Applying  $I_w \otimes_\Lambda ?$  and using Theorem 5.3 (iv), we conclude that  $Z = I_w \otimes_\Lambda \text{Hom}_\Lambda(I_w, Z)$  contains  $T^{ws_i} = I_w \otimes_\Lambda \Sigma_i^* \Sigma_i X$ .  $\square$

This proposition admits a dual version, which we state for reference.

**Theorem 5.12** (i) *Let  $(w, i) \in W \times I$  be such that  $\ell(s_i w) > \ell(w)$  and let  $T \in \Lambda\text{-mod}$ . Then  $T_{s_i w}/T_w$  is the largest submodule of  $T/T_w$  isomorphic to a direct sum of copies of the module  $\text{Hom}_\Lambda(I_w, S_i)$ .*

(ii) *Let  $w = s_{i_\ell} \cdots s_{i_1}$  be a reduced decomposition. Then a module  $T \in \Lambda\text{-mod}$  belongs to  $\mathcal{T}_w$  if and only if it has a filtration whose subquotients are modules of the form  $\text{Hom}_\Lambda(I_{s_{i_{k-1}} \cdots s_{i_1}}, S_{i_k})$  with  $1 \leq k \leq \ell$ .*

(iii) *Let  $w = s_{i_\ell} \cdots s_{i_1}$  be a reduced decomposition. Then a module  $T \in \Lambda\text{-mod}$  belongs to  $\mathcal{F}_w$  if and only if there is no monomorphism  $\text{Hom}_\Lambda(I_{s_{i_{k-1}} \cdots s_{i_1}}, S_{i_k}) \hookrightarrow T$  with  $1 \leq k \leq \ell$ .*

As already mentioned, most of the results presented above in this section can be found in papers by Iyama, Reiten and al. and by Geiß, Leclerc and Schröer. The aim of the next three examples is to explain some connections in more detail.

*Example 5.13.* Let us denote by  $\{\omega_i \mid i \in I\}$  the set of fundamental weights of the root system  $\Phi$ ; these weights are elements of a representation of  $W$  that extends  $\mathbb{R}I$ , and for all  $(i, j) \in I^2$ , we have  $s_j \omega_i = \omega_i - \delta_{ij} \alpha_i$ , where  $\delta_{ij}$  is Kronecker's symbol. Let us now fix  $(i, w) \in I \times W$ . By [5], Theorem 3.1 (ii), there exists a unique  $\Lambda$ -module  $N(-w\omega_i)$  whose dimension-vector is  $\omega_i - w\omega_i$  and whose socle is 0 or  $S_i$ . (The paper [5] mainly deals with the case where  $\mathfrak{g}$  is finite dimensional, but the constructions in Sections 2 and 3 are valid

in general, with the exception of Proposition 3.6.) The aim of this example is to show that  $N(-w\omega_i) \cong ((\Lambda/I_w) \otimes_\Lambda \Lambda e_i)^*$ . For that, we consider a reduced decomposition  $w = s_{j_1} \cdots s_{j_\ell}$ . The filtration  $\Lambda \supset I_{s_{j_1}} \supset I_{s_{j_1}s_{j_2}} \supset \cdots \supset I_{s_{j_1}\cdots s_{j_{\ell-1}}} \supset I_w$  induces a filtration of  $\Lambda/I_w$ , whose  $k$ -th subquotient is the layer

$$I_{s_{j_1}\cdots s_{j_{k-1}}}/I_{s_{j_1}\cdots s_{j_k}} \cong I_{s_{j_1}\cdots s_{j_{k-1}}} \otimes_\Lambda (\Lambda/I_{j_k}) = I_{s_{j_1}\cdots s_{j_{k-1}}} \otimes_\Lambda S_{j_k}$$

studied by Amiot, Iyama, Reiten and Todorov [1]. Tensoring this filtration on the right with the projective  $\Lambda$ -module  $\Lambda e_i$  kills all the subquotients with  $j_k \neq i$ , so by Lemma 5.10,

$$\underline{\dim} (\Lambda/I_w) \otimes_\Lambda \Lambda e_i = \sum_{\substack{1 \leq k \leq \ell \\ j_k = i}} s_{j_1} \cdots s_{j_{k-1}} \alpha_{j_k} = \omega_i - w\omega_i.$$

In addition, the head of  $\Lambda e_i$  is equal to  $S_i$ , so the head of  $(\Lambda/I_w) \otimes_\Lambda \Lambda e_i$  is 0 or  $S_i$ . The dual of  $(\Lambda/I_w) \otimes_\Lambda \Lambda e_i$  therefore satisfies the two conditions that characterize  $N(-w\omega_i)$ .

*Example 5.14.* Theorem 5.11 (ii) and Corollary 2.9 in [29] imply that our category  $\mathcal{F}^w$  is equal to  $\text{Sub}(\Lambda/I_w)$ , the full subcategory of  $\Lambda\text{-mod}$  whose objects are the modules that can be embedded in a direct sum of copies of  $\Lambda/I_w$ . (To make sure that the assumptions of the statements in [29] are fulfilled, observe that a module  $T \in \mathcal{F}^w$  satisfies  $\text{Hom}_\Lambda(I_w, T) = 0$  by definition, whence  $I_w T = 0$ , so  $T$  can be seen as a  $\Lambda/I_w$ -module.)

*Example 5.15.* We now compare Theorem 5.11 to Geiß, Leclerc and Schröer's stratification. We fix a reduced expression  $w = s_{i_\ell} \cdots s_{i_1}$ . Then Geiß, Leclerc and Schröer define modules  $V_k$  and  $M_k$  for  $1 \leq k \leq \ell$  (Sections 2.4 and 9 of [24]). By construction,  $V_k$  is a submodule of the injective hull of  $S_{i_k}$ , hence has socle 0 or  $S_{i_k}$ ; moreover, the dimension-vector of  $V_k$  is  $\omega_{i_k} - s_{i_1} \cdots s_{i_k} \omega_{i_k}$ , by Corollary 9.2 in [24]. Comparing with Example 5.13, we see that

$$V_k \cong N(-s_{i_1} \cdots s_{i_k} \omega_{i_k}) \cong ((\Lambda/I_{s_{i_1}\cdots s_{i_k}}) \otimes_\Lambda \Lambda e_{i_k})^*.$$

Likewise, with  $k^- = \max(\{0\} \cup \{s \in \{1, \dots, k-1\} \mid i_s = i_k\})$ , we have

$$V_{k^-} \cong N(-s_{i_1} \cdots s_{i_{k^-}} \omega_{i_{k^-}}) = N(-s_{i_1} \cdots s_{i_{k-1}} \omega_{i_k}) \cong ((\Lambda/I_{s_{i_1}\cdots s_{i_{k-1}}}) \otimes_\Lambda \Lambda e_{i_k})^*.$$

The quotient  $M_k = V_k/V_{k^-}$  is thus the dual of

$$(I_{s_{i_1}\cdots s_{i_{k-1}}}/I_{s_{i_1}\cdots s_{i_k}}) \otimes_\Lambda \Lambda e_{i_k} \cong I_{s_{i_1}\cdots s_{i_{k-1}}} \otimes_\Lambda (\Lambda/I_{i_k}) \otimes_\Lambda \Lambda e_{i_k} \cong I_{s_{i_1}\cdots s_{i_{k-1}}} \otimes_\Lambda S_{i_k}.$$

By Example 5.6 (iii), this gives  $M_k \cong \text{Hom}_\Lambda(I_{s_{i_{k-1}}\cdots s_{i_1}}, S_{i_k})$ . In view of Lemma 10.2 in [24], we can therefore identify the category  $\mathcal{C}_w$  of [24] with our category  $\mathcal{T}_w$ , and on a  $\Lambda$ -module  $T \in \mathcal{T}_w$ ,

we can match the stratification constructed in Section 10 of [24] with the filtration by the submodules  $T_{s_{i_k} \dots s_{i_1}}$ . Comparing with the conclusion of Example 5.14, we additionally obtain that  $\mathcal{T}_w = \text{Fac}((\Lambda/I_{w-1})^*)$ , the full subcategory of  $\Lambda\text{-mod}$  whose objects are the homomorphic images of a direct sum of copies of  $(\Lambda/I_{w-1})^*$ . This is in agreement with Theorem 2.8 (iv) in [24].

Theorem 5.11 has the following noteworthy consequence.

**Proposition 5.16** *Let  $(u, v) \in W^2$  such that  $\ell(u) + \ell(v) = \ell(uv)$ . Then  $(\mathcal{T}_u, \mathcal{F}_u) \preceq (\mathcal{T}^v, \mathcal{F}^v)$ .*

*Proof.* We want to show that  $\mathcal{T}_u \cap \mathcal{F}^v = \{0\}$ . Assume the contrary and choose  $T \neq 0$  of minimal dimension in the intersection. Write  $u = s_{i_\ell} \dots s_{i_1}$  and  $v = s_{j_1} \dots s_{j_m}$ . By Theorem 5.11 (ii),  $T$  has a filtration with subquotients of the form  $I_{s_{j_1} \dots s_{j_{q-1}}} \otimes_\Lambda S_{j_q}$ . Any subquotient of this filtration belongs to  $\mathcal{F}^v$ , and the top one also belongs to  $\mathcal{T}_u$  since a torsion class is closed under taking quotients. The minimality of  $\dim T$  imposes then that the filtration has just one step. Dually,  $T$  has a filtration with subquotients of the form  $\text{Hom}_\Lambda(I_{s_{i_{p-1}} \dots s_{i_1}}, S_{i_p})$ , and minimality impose again that this filtration has just one step. We end up with an isomorphism

$$\text{Hom}_\Lambda(I_{s_{i_{p-1}} \dots s_{i_1}}, S_{i_p}) \cong I_{s_{j_1} \dots s_{j_{q-1}}} \otimes_\Lambda S_{j_q}.$$

Taking dimension-vectors, we get  $s_{i_1} \dots s_{i_{p-1}} \alpha_{i_p} = s_{j_1} \dots s_{j_{q-1}} \alpha_{j_q}$ , by Lemma 5.10. This contradicts the assumption  $\ell(u) + \ell(v) = \ell(uv)$ .  $\square$

We conclude this section with a proposition that slightly refines Proposition 3.2 in [29].

**Proposition 5.17** *Assume that  $\ell(u) + \ell(v) = \ell(uv)$ . Then one has equivalences of categories*

$$\mathcal{F}^v \xrightleftharpoons[\text{Hom}_\Lambda(I_u, ?)]{I_u \otimes_\Lambda ?} \mathcal{F}^{uv} \cap \mathcal{T}^u \quad \text{and} \quad \mathcal{T}_{uv} \cap \mathcal{F}^v \xrightleftharpoons[\text{Hom}_\Lambda(I_v, ?)]{I_v \otimes_\Lambda ?} \mathcal{T}_u.$$

*Proof.* If  $T \in \mathcal{F}^v$ , then  $T \in \mathcal{F}_u$ , by Proposition 5.16. Therefore  $T = T_u \cong \text{Hom}_\Lambda(I_u, I_u \otimes_\Lambda T)$ , so

$$\text{Hom}_\Lambda(I_{uv}, I_u \otimes_\Lambda T) \cong \text{Hom}_\Lambda(I_v, \text{Hom}_\Lambda(I_u, I_u \otimes_\Lambda T)) \cong \text{Hom}_\Lambda(I_v, T) = 0,$$

which shows that  $I_u \otimes_\Lambda T \in \mathcal{F}^{uv}$ . Conversely, if  $T \in \mathcal{F}^{uv}$ , then

$$\text{Hom}_\Lambda(I_v, \text{Hom}_\Lambda(I_u, T)) = \text{Hom}_\Lambda(I_{uv}, T) = 0,$$

hence  $\text{Hom}_\Lambda(I_u, T) \in \mathcal{F}^v$ . The first pair of equivalences then follows from Theorem 5.4 (iii).

The proof of the second equivalence is similar.  $\square$



## 5.4 Tilting structure and HN polytopes in $\Lambda\text{-mod}$

We now relate all this material about the reflection functors and the torsion pairs  $(\mathcal{T}^w, \mathcal{F}^w)$  and  $(\mathcal{T}_w, \mathcal{F}_w)$  to the categories  $\mathcal{I}_\theta$ ,  $\mathcal{R}_\theta$ , etc., and to the HN polytopes defined in Section 3.

The following result is almost identical to Theorem 4.1 in [52]; we just state it in a more general way and prove the key point in a different fashion.

**Theorem 5.18** *Let  $\theta : \mathbb{Z}I \rightarrow \mathbb{R}$  be a group homomorphism and let  $i \in I$ . If  $\langle \theta, \alpha_i \rangle > 0$ , then  $\Sigma_i$  and  $\Sigma_i^*$  induce mutually inverse equivalences*

$$\mathcal{R}_\theta \xrightleftharpoons[\Sigma_i]{\Sigma_i^*} \mathcal{R}_{s_i\theta}.$$

*Proof.* The assumption  $\langle \theta, \alpha_i \rangle > 0$  forbids to a module  $T \in \mathcal{R}_\theta$  to have a submodule isomorphic to  $S_i$ , so  $\mathcal{R}_\theta \subseteq \mathcal{F}_{s_i}$  by Example 5.6 (i). Likewise,  $\mathcal{R}_{s_i\theta} \subseteq \mathcal{T}^{s_i}$ .

Let  $T \in \mathcal{R}_\theta$ . Then  $T \in \mathcal{F}_{s_i}$  and  $\langle \theta, \underline{\dim} T \rangle = 0$ . Using Corollary 5.8 with  $w = s_i$ , we obtain  $\langle s_i\theta, \underline{\dim} \Sigma_i^* T \rangle = 0$ .

To show that  $\Sigma_i^* T$  is  $s_i\theta$ -semistable, it remains to show that  $\langle s_i\theta, \underline{\dim} X \rangle \geq 0$  for any quotient  $X$  of  $\Sigma_i^* T$ . In this aim, consider a surjective morphism  $f : \Sigma_i^* T \rightarrow X$ . The functor  $\Sigma_i$  modify just the vector space at vertex  $i$ , so the default of surjectivity of  $\Sigma_i f$  is concentrated at this vertex. There exists thus a natural integer  $n$  such that  $\underline{\dim} \text{coker}(\Sigma_i f) = n\alpha_i$ . Note now that not only  $\Sigma_i^* T$ , but also  $X$  belong to  $\mathcal{T}^{s_i}$ , for a torsion class is closed under taking quotients. By Corollary 5.8 again, we have  $\underline{\dim} \Sigma_i X = s_i \underline{\dim} X$ . We therefore have  $s_i \underline{\dim} X = \underline{\dim} \text{im}(\Sigma_i f) + n\alpha_i$ . Since  $T$  is  $\theta$ -semistable,  $\langle \theta, \underline{\dim} \text{im}(\Sigma_i f) \rangle \geq 0$ . We eventually find that  $\langle s_i\theta, \underline{\dim} X \rangle \geq n\langle \theta, \alpha_i \rangle \geq 0$ , as desired.

We thus see that  $\Sigma_i^*$  maps  $\mathcal{R}_\theta$  to  $\mathcal{R}_{s_i\theta}$ . A dual reasoning shows that  $\Sigma_i$  maps  $\mathcal{R}_{s_i\theta}$  to  $\mathcal{R}_\theta$ . The theorem now follows from Theorem 5.4 (iii) and from the inclusions  $\mathcal{R}_\theta \subseteq \mathcal{F}_{s_i}$  and  $\mathcal{R}_{s_i\theta} \subseteq \mathcal{T}^{s_i}$ .  $\square$

Recall from Section 2.1 the definition of the dominant cone  $\overline{C}_0$  and of the Tits cone  $C_T = \bigcup_{w \in W} w\overline{C}_0$ . A subset  $J \subseteq I$  gives rise to a face  $F_J \subseteq \overline{C}_0$ , to a parabolic subgroup  $W_J = \langle s_j \mid j \in J \rangle$  of  $W$ , and to a root system  $\Phi_J \subseteq \Phi$ . In the case  $W_J$  is finite, we denote its longest element by  $w_J$ . Recall also that for  $w \in W$ , we set  $N_w = \Phi_+ \cap w\Phi_-$  and that for any reduced decomposition  $w = s_{i_1} \cdots s_{i_\ell}$ , we have

$$N_w = \{s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \mid 1 \leq k \leq \ell\}.$$

**Theorem 5.19** *Let  $J \subseteq I$ , let  $\theta \in F_J$ , and let  $w \in W$ . We assume that  $w$  is  $J$ -reduced on the right, that is,  $\ell(ws_j) > \ell(w)$  for each  $j \in J$ .*

(i) *The category  $\mathcal{R}_\theta$  coincides with the subcategory  $\Lambda_J\text{-mod}$ . Moreover, there are mutually inverse equivalences*

$$\mathcal{R}_\theta \xrightleftharpoons[\text{Hom}_\Lambda(I_w, ?)]{I_w \otimes_\Lambda ?} \mathcal{R}_{w\theta}.$$

(ii) *We have  $(\mathcal{T}^w, \mathcal{F}^w) = (\overline{\mathcal{T}}_{w\theta}, \mathcal{P}_{w\theta})$ .*

(iii) *If  $W_J$  is finite, then have  $(\mathcal{T}^{ww_J}, \mathcal{F}^{ww_J}) = (\mathcal{T}_{w\theta}, \overline{\mathcal{P}}_{w\theta})$ .*

*Proof.* Let  $J, \theta, w$  as in the statement of the theorem.

Given  $T \in \Lambda\text{-mod}$ , the condition  $\langle \theta, \underline{\dim} T \rangle = 0$  is necessary for  $T$  to be in  $\mathcal{R}_\theta$ . It is also sufficient, because any quotient module  $X$  of  $T$  satisfies  $\langle \theta, \underline{\dim} T \rangle \geq 0$  by the dominance of  $\theta$ . The equality  $\mathcal{R}_\theta = \Lambda_J\text{-mod}$  then follows from Example 5.6 (ii).

Since  $\theta$  is in  $F_J$ , it takes a positive value at each root in  $\Phi_+ \setminus \Phi_J$ . Lemma 2.2 then ensures that  $\theta$  take positive values on  $N_{w^{-1}}$ . Choosing a reduced decomposition  $w = s_{i_1} \cdots s_{i_\ell}$ , we obtain  $\langle s_{i_{k+1}} \cdots s_{i_\ell} \theta, \alpha_{i_k} \rangle > 0$  for each  $1 \leq k \leq \ell$ . Using Theorem 5.18, we get a chain of equivalences of categories

$$\mathcal{R}_\theta \xrightleftharpoons[\Sigma_{i_\ell}]{\Sigma_{i_\ell}^*} \mathcal{R}_{s_{i_\ell} \theta} \xrightleftharpoons[\Sigma_{i_{\ell-1}}]{\Sigma_{i_{\ell-1}}^*} \cdots \xrightleftharpoons[\Sigma_{i_2}]{\Sigma_{i_2}^*} \mathcal{R}_{s_{i_\ell} \cdots s_{i_2} \theta} \xrightleftharpoons[\Sigma_{i_1}]{\Sigma_{i_1}^*} \mathcal{R}_{w\theta}.$$

By composition, we get assertion (i).

Let  $T \in \mathcal{T}^w$  and let  $X$  be a quotient of  $T$ ; then  $X \in \mathcal{T}^w$ . By Theorem 5.4 (iii) and Corollary 5.8,  $\underline{\dim} X$  is of the form  $w\nu$  with  $\nu \in \mathbb{N}I$ , and so  $\langle w\theta, \underline{\dim} X \rangle = \langle \theta, \nu \rangle \geq 0$ . This proves that  $T \in \overline{\mathcal{T}}_{w\theta}$ .

Let  $T \in \mathcal{F}^w$  and let  $X \subseteq T$  be a nonzero submodule; then  $X$  is in  $\mathcal{F}^w$ . Theorem 5.11 (ii) and Lemma 5.10 then imply that  $\underline{\dim} X$  is a nontrivial  $\mathbb{N}$ -linear combination of elements in  $N_w$ . In addition,  $w\theta$  takes negative values on  $N_w$ , for  $\theta$  takes positive values on  $N_{w^{-1}} = -w^{-1}N_w$ , as we have seen during the course of the proof of assertion (i). Therefore  $\langle w\theta, \underline{\dim} X \rangle < 0$ . This reasoning shows that  $T \in \mathcal{P}_{w\theta}$ .

We have established that  $\mathcal{T}^w \subseteq \overline{\mathcal{P}}_{w\theta}$  and that  $\mathcal{F}^w \subseteq \mathcal{P}_{w\theta}$ . This implies assertion (ii).

We now prove assertion (iii), assuming that  $W_J$  is finite.

Consider first  $T \in \mathcal{T}^{ww_J}$  and take a nonzero quotient  $X$  of  $T$ . Then  $X$  also belongs to  $\mathcal{T}^{ww_J}$ , and by Proposition 5.7, we can write  $X = I_w \otimes_{\Lambda} Y$ , with  $Y \in \mathcal{T}^{w_J}$ . Since  $X \neq 0$ , we have  $Y \neq 0$ , hence  $Y \notin \mathcal{F}^{w_J}$ . By Example 5.6 (ii), this means that  $\underline{\dim} Y$  is not in  $\mathbb{N}J$ , the set of  $\mathbb{N}$ -linear combinations of elements in  $\{\alpha_j \mid j \in J\}$ . Therefore  $\langle w\theta, \underline{\dim} X \rangle = \langle \theta, \underline{\dim} Y \rangle > 0$ . This proves that  $T \in \mathcal{J}_{w\theta}$ .

Now let  $T \in \mathcal{F}^{ww_J}$  and take a submodule  $X \subseteq T$ . Then  $X \in \mathcal{F}^{ww_J}$  as well. Theorem 5.11 (ii) and Lemma 5.10 then imply that  $\underline{\dim} X$  is a  $\mathbb{N}$ -linear combination of elements in  $N_{ww_J}$ . Certainly,  $ww_J\theta$  takes nonpositive values on  $N_{ww_J}$ , so  $\langle ww_J\theta, \underline{\dim} X \rangle \leq 0$ . Observing that  $w_J\theta = \theta$ , we conclude that  $T \in \overline{\mathcal{J}}_{w\theta}$ .

We have established that  $\mathcal{T}^{ww_J} \subseteq \mathcal{P}_{w\theta}$  and that  $\mathcal{F}^{ww_J} \subseteq \overline{\mathcal{J}}_{w\theta}$ , whence assertion (iii).  $\square$

*Remarks 5.20.* (i) In the context of assertions (ii) and (iii) of Theorem 5.19, we have  $T^w = T_{w\theta}^{\max}$  and  $T^{ww_J} = T_{w\theta}^{\min}$  for any  $\Lambda$ -module  $T$ . This shows in particular that each  $\underline{\dim} T^w$  is a vertex of the HN polytope  $\text{Pol}(T)$ .

(ii) Theorem 5.19 admits a dual version, which can be obtained with the help of Remark 4.1 and Example 5.6 (iii).

(iii) Let us choose  $\theta$  in the open dominant cone  $C_0$ . Then for each  $w \in W$ , we have  $\mathcal{T}_w = \mathcal{J}_{-w^{-1}\theta}$ . Remark 14.2 in [24] becomes then a particular case of our Proposition 4.2 (iii).

(iv) In the case where  $W_J$  is finite, assertion (i) of Theorem 5.19 is a particular case of Proposition 5.17, since  $\mathcal{R}_{w\theta} = \overline{\mathcal{P}}_{w\theta} \cap \overline{\mathcal{P}}_{w\theta} = \mathcal{T}^w \cap \mathcal{F}^{ww_J}$  and  $\mathcal{R}_{\theta} = \Lambda_J\text{-mod} = \mathcal{F}^{w_J}$ .

Our first corollary to Theorem 5.19 shows that the position of the facets of our polytopes can be computed as the dimension of homomorphism spaces.

**Corollary 5.21** *Let  $J$ ,  $\theta$  and  $w$  be as in Theorem 5.19. Suppose that  $\theta$  is integral, that is, each  $\langle \theta, \alpha_i \rangle$  is an integer. Set  $N(w\theta) = \bigoplus_{i \in I} (I_w \otimes_{\Lambda} \Lambda e_i)^{\oplus \langle \theta, \alpha_i \rangle}$  and denote by  $N(-w\theta)$  the dual of  $\bigoplus_{i \in I} ((\Lambda/I_w) \otimes_{\Lambda} \Lambda e_i)^{\oplus \langle \theta, \alpha_i \rangle}$ . Then for any  $\Lambda$ -module  $T$ ,*

$$\dim \text{Hom}_{\Lambda}(N(\pm w\theta), T) = \psi_{\text{Pol}(T)}(\pm w\theta).$$

*Proof.* The case  $+w\theta$  comes from the computation

$$\begin{aligned}
\dim \operatorname{Hom}_\Lambda(N(w\theta), T) &= \sum_{i \in I} \langle \theta, \alpha_i \rangle \dim \operatorname{Hom}_\Lambda(\Lambda e_i, \operatorname{Hom}_\Lambda(I_w, T)) \\
&= \sum_{i \in I} \langle \theta, \alpha_i \rangle (\operatorname{Hom}_\Lambda(I_w, T) : S_i) \\
&= \langle \theta, \underline{\dim} \operatorname{Hom}_\Lambda(I_w, T) \rangle \\
&= \langle w\theta, \underline{\dim} T^w \rangle \\
&= \langle w\theta, \underline{\dim} T_{w\theta}^{\max} \rangle \\
&= \psi_{\operatorname{Pol}(T)}(w\theta).
\end{aligned}$$

Here the first equality is adjunction, the second uses that  $\Lambda e_i$  is the projective cover of the simple  $\Lambda$ -module  $S_i$ , the fourth is Corollary 5.8, and the fifth is Theorem 5.19 (ii).

Now applying the functor  $\operatorname{Hom}_\Lambda(?, T^*)$  to the short exact sequence  $0 \rightarrow I_w \otimes_\Lambda \Lambda e_i \rightarrow \Lambda e_i \rightarrow (\Lambda/I_w) \otimes_\Lambda \Lambda e_i \rightarrow 0$ , we get a long exact sequence

$$\begin{aligned}
0 \rightarrow \operatorname{Hom}_\Lambda((\Lambda/I_w) \otimes_\Lambda \Lambda e_i, T^*) &\rightarrow \operatorname{Hom}_\Lambda(\Lambda e_i, T^*) \\
&\rightarrow \operatorname{Hom}_\Lambda(I_w \otimes_\Lambda \Lambda e_i, T^*) \rightarrow \operatorname{Ext}_\Lambda^1((\Lambda/I_w) \otimes_\Lambda \Lambda e_i, T^*) \rightarrow 0.
\end{aligned}$$

Taking dimensions, and using Crawley-Boevey's formula (4.2) and Example 5.13, we obtain

$$\begin{aligned}
&\dim \operatorname{Hom}_\Lambda(T^*, (\Lambda/I_w) \otimes_\Lambda \Lambda e_i) \\
&= \dim \operatorname{Hom}_\Lambda(I_w \otimes_\Lambda \Lambda e_i, T^*) - \dim \operatorname{Hom}_\Lambda(\Lambda e_i, T^*) + (\underline{\dim} (\Lambda/I_w) \otimes_\Lambda \Lambda e_i, \underline{\dim} T^*) \\
&= \dim \operatorname{Hom}_\Lambda(I_w \otimes_\Lambda \Lambda e_i, T^*) - (T^* : S_i) + (\omega_i - w\omega_i, \underline{\dim} T^*).
\end{aligned}$$

Multiplying by  $\langle \theta, \alpha_i \rangle$  and summing over  $I$  gives then

$$\begin{aligned}
\dim \operatorname{Hom}_\Lambda(N(-w\theta), T) &= \sum_{i \in I} \langle \theta, \alpha_i \rangle \dim \operatorname{Hom}_\Lambda(T^*, (\Lambda/I_w) \otimes_\Lambda \Lambda e_i) \\
&= \dim \operatorname{Hom}_\Lambda(N(w\theta), T^*) - \langle \theta, \underline{\dim} T^* \rangle + \langle \theta - w\theta, \underline{\dim} T^* \rangle \\
&= \psi_{\operatorname{Pol}(T^*)}(w\theta) - \langle w\theta, \underline{\dim} T \rangle.
\end{aligned}$$

Remark 4.1 shows that the right-hand side is  $\psi_{\operatorname{Pol}(T)}(-w\theta)$ .  $\square$

Our second corollary compares the normal fan of an HN polytope to the Tits fan.

**Corollary 5.22** *The support function of the HN polytope of a  $\Lambda$ -module is linear on each face  $wF_J$  of the Tits cone.*

*Proof.* Let  $T$  be a  $\Lambda$ -module. The support function of  $\text{Pol}(T)$  is given by  $\theta \mapsto \langle \theta, \underline{\dim} T_\theta^{\max} \rangle$ . However,  $T_\theta^{\max} = T^w$  if  $\theta \in wF_J$ , with  $w$  chosen  $J$ -reduced on the right.  $\square$

Using Remark 5.20 (ii), we see that the support function of an HN polytope is also linear on each face of the opposite  $-C_T$  of the Tits cone.

The third corollary describe all the faces of an HN polytope defined by a linear form in the Tits cone.

**Corollary 5.23** *Let  $J$ ,  $\theta$  and  $w$  as in the statement of Theorem 5.19 and let  $T \in \Lambda\text{-mod}$ . Set  $X = \text{Hom}_\Lambda(I_w, T_{w\theta}^{\max}/T_{w\theta}^{\min})$ . Then  $X$  is in the subcategory  $\Lambda_J\text{-mod}$  and*

$$\{x \in \text{Pol}(T) \mid \langle w\theta, x \rangle = \psi_{\text{Pol}(T)}(w\theta)\} = \underline{\dim} T_{w\theta}^{\min} + w \text{Pol}(X).$$

*Proof.* This follows by combining Corollary 3.3, Theorem 5.19 (i), Corollary 5.8 and Example 5.6 (ii).  $\square$

In the case  $J = \emptyset$ , that is, if  $\theta$  belongs to the open cone  $C_0$ , then  $T^w = T_{w\theta}^{\min} = T_{w\theta}^{\max}$ ,  $\mathcal{I}_{w\theta} = \overline{\mathcal{I}}_{w\theta} = \mathcal{I}^w$ ,  $\mathcal{P}_{w\theta} = \overline{\mathcal{P}}_{w\theta} = \mathcal{F}^w$ , and  $\mathcal{R}_{w\theta} \cong \mathcal{R}_\theta = \Lambda_J\text{-mod} = \{0\}$ . This case corresponds of course to a vertex of  $\text{Pol}(T)$ .

In the case where  $J$  has just one element, say  $i$ , then  $w_J = s_i$ ,  $(T_{w\theta}^{\min}, T_{w\theta}^{\max}) = (T^{ws_i}, T^w)$ ,  $\mathcal{R}_\theta = \Lambda_J\text{-mod} = \text{add } S_i$ , and  $\mathcal{R}_{w\theta} = \text{add}(I_w \otimes_\Lambda S_i)$ . This case corresponds to an edge of  $\text{Pol}(T)$  that points in the direction  $\underline{\dim}(I_w \otimes_\Lambda S_i) = w\alpha_i$ .

The case where  $J$  contains just two vertices  $i$  and  $j$  linked with a single edge is more interesting. Here  $w_J = s_i s_j s_i$ ,  $(T_{w\theta}^{\min}, T_{w\theta}^{\max}) = (T^{ww_J}, T^w)$ , and  $\mathcal{R}_\theta = \Lambda_J\text{-mod}$  has four indecomposables. This case corresponds to a 2-face of type  $A_2$ . We will come back to this case soon: if  $T$  is a general point in an irreducible component  $\Lambda_b$ , then this 2-face will be constrained by the tropical Plücker relations.

## 5.5 Tilting structure and crystal operations

Thanks to Theorem 4.4, we know that under a suitable openness condition (O), each torsion pair  $(\mathcal{T}, \mathcal{F})$  gives rise to a bijection  $\Xi : \mathfrak{T} \times \mathfrak{F} \rightarrow \mathfrak{B}$ . The aim of this section is to show that in the case of the torsion pair  $(\mathcal{T}^w, \mathcal{F}^w)$ , this bijection can be described by elementary operations on the crystal  $\mathfrak{B}$ . Given  $\nu \in \text{NI}$ , both sets  $\{T \in \Lambda(\nu) \mid T \in \mathcal{T}^w\}$  and  $\{T \in \Lambda(\nu) \mid T \in \mathcal{F}^w\}$  are open (Proposition 4.2 (iii) and Theorem 5.19 (ii)), so we can define the subsets  $\mathfrak{T}^w$  and  $\mathfrak{F}^w$  of  $\mathfrak{B}$  formed by irreducible components whose general point belongs to  $\mathcal{T}^w$  and  $\mathcal{F}^w$ . We define  $\mathfrak{T}_w$  and  $\mathfrak{F}_w$  in a similar fashion.

The first ingredient is the particular case where  $w$  is a simple reflection  $s_i$ . By Example 5.6 (i),  $\mathfrak{F}^{s_i} = \bigsqcup_{n \in \mathbb{N}} \mathfrak{B}_{n\alpha_i}$  is in bijection with  $\mathbb{N}$ ; moreover, by the definition of the crystal structure on  $\mathfrak{B}$  (see Section 4.3), we have  $\mathfrak{T}^{s_i} = \{Z \in \mathfrak{B} \mid \varphi_i(Z) = 0\}$ . Under the identification  $\mathfrak{F}^{s_i} \cong \mathbb{N}$ , the bijection  $\mathfrak{T}^{s_i} \times \mathfrak{F}^{s_i} \rightarrow \mathfrak{B}$  becomes the map  $(Z, n) \mapsto \tilde{e}_i^n Z$ . The inverse of this map is  $Z \mapsto (\tilde{f}_i^{\max} Z, \varphi_i(Z))$ .

To go further, we need to understand how the equivalence of categories provided by Theorem 5.4 (iii) relates to crystal operations. Recall that a key feature of  $B(-\infty)$  is the existence of a weight-preserving involution, denoted by  $*$  (see [33], §8.3). One can then define the starred operators  $\tilde{e}_i^* = (b \mapsto (\tilde{e}_i b^*)^*)$  and  $\tilde{f}_i^* = (b \mapsto (\tilde{f}_i b^*)^*)$  on  $\mathfrak{B}$ . In [50], Corollary 3.4.8 (see also [34], Section 8.2), Saito defines mutually inverse bijections

$$\{b \in B(-\infty) \mid \varphi_i(b) = 0\} \xrightleftharpoons[S_i^*]{S_i} \{b \in B(-\infty) \mid \varphi_i(b^*) = 0\}$$

by the rules  $S_i(b) = \tilde{e}_i^{\varepsilon_i(b^*)}(\tilde{f}_i^*)^{\max} b$  and  $S_i^*(b) = (\tilde{e}_i^*)^{\varepsilon_i(b)} \tilde{f}_i^{\max} b$ . This definition ensures that if  $\varphi_i(b) = 0$ , then  $\text{wt } S_i(b) = s_i(\text{wt } b)$ . For convenience, we extend  $S_i$  and  $S_i^*$  on  $B(-\infty)$  by setting  $\sigma_i b = S_i(\tilde{f}_i^{\max} b)$  and  $\sigma_i^* b = S_i^*((\tilde{f}_i^*)^{\max} b)$ . By transport through the bijection  $B(-\infty) \cong \mathfrak{B}$ , we can view the maps  $\sigma_i$  and  $\sigma_i^*$  as maps from  $\mathfrak{B}$  to itself. In view of Example 5.6 (i),  $\sigma_i$  and  $\sigma_i^*$  restrict to mutually inverse bijections

$$\mathfrak{F}_{s_i} \xrightleftharpoons[\sigma_i]{\sigma_i^*} \mathfrak{T}_{s_i}$$

**Proposition 5.24** (i) *Let  $\nu \in \mathbb{N}I$ , let  $i \in I$ , let  $Z \in \mathfrak{F}_{s_i}(\nu)$  and let  $Z' = \sigma_i^*(Z)$ , an element in  $\mathfrak{T}^{s_i}(s_i\nu)$ . Let  $U = \{T \in Z \mid \text{soc}_i T = 0\}$  and  $U' = \{T' \in Z' \mid \text{hd}_i T' = 0\}$ . Let  $\Theta$  be the set of triples  $(T, T', h)$ , such that  $(T, T') \in U \times U'$  and  $h : T' \rightarrow \Sigma_i^* T$  is an isomorphism. Then the first projection  $\Theta \rightarrow U$  and the second one  $\Theta \rightarrow U'$  are locally trivial fibrations with a smooth and connected fiber.*

(ii) *Let  $(u, v, i) \in W^2 \times I$  be such that  $\ell(us_i w) = \ell(u) + \ell(w) + 1$ . Then  $\mathfrak{F}^v \subseteq \mathfrak{F}_{us_i}$  and  $\mathfrak{F}^{s_i v} \subseteq \mathfrak{F}_u$ . In addition,  $\sigma_i$  and  $\sigma_i^*$  restrict to mutually inverse bijections*

$$\mathfrak{F}_u \cap \mathfrak{T}^{s_i w} \xrightleftharpoons[\sigma_i^*]{\sigma_i} \mathfrak{F}_{us_i} \cap \mathfrak{T}^w \quad \text{and} \quad \mathfrak{F}^{s_i w} \cap \mathfrak{T}^{s_i} \xrightleftharpoons[\sigma_i^*]{\sigma_i} \mathfrak{F}^w.$$

*Proof.* Assertion (i) is [5], Theorem 5.3. It is the precise way of stating that if  $T$  is a general point in  $Z$ , then  $\Sigma_i^*(T)$  “belongs” to  $Z$  and is general in  $Z$ . (The quotes around “belongs” reflects the fact that a point of  $Z'$  can only be isomorphic to  $\Sigma_i(T)$ , and not equal to it.) Assertion (ii) follows then from Propositions 5.7 and 5.17.  $\square$

Let  $(w, i) \in W \times I$  be such that  $\ell(ws_i) > \ell(w)$ . By Theorem 5.11 (i),  $\mathcal{F}^{ws_i} \cap \mathcal{T}^w = \text{add}(I_w \otimes_{\Lambda} S_i)$ , so  $\mathfrak{F}^{ws_i} \cap \mathfrak{T}^w$  is in bijection with  $\mathbb{N}$ : to an integer  $n$  corresponds the closure in  $\Lambda(w\alpha_i)$  of the orbit representing the module  $I_w \otimes_{\Lambda} S_i^{\oplus n}$ .

Now choose a finite sequence  $\mathbf{i} = (i_1, \dots, i_{\ell})$  such that  $w = s_{i_1} \cdots s_{i_{\ell}}$  is a reduced decomposition. For  $0 \leq k \leq \ell$ , set  $(\mathcal{T}_k, \mathcal{F}_k) = (\mathcal{T}^{s_{i_1} \cdots s_{i_k}}, \mathcal{F}^{s_{i_1} \cdots s_{i_k}})$ . Then

$$(\mathcal{T}_{\ell}, \mathcal{F}_{\ell}) \preceq \cdots \preceq (\mathcal{T}_1, \mathcal{F}_1).$$

The generalization of Proposition 4.6 to finite sequences provides a bijection

$$\Omega_{\mathbf{i}} : \mathfrak{B} \rightarrow \prod_{k=1}^{\ell} (\mathfrak{F}_k \cap \mathfrak{T}_{k-1}) \times \mathfrak{T}_{\ell}.$$

(Here we have used the inverse map to that defined in Proposition 4.6 and have reversed the order of the factors.) Under the identification  $\mathfrak{F}_k \cap \mathfrak{T}_{k-1} \cong \mathbb{N}$ , this bijection  $\Omega_{\mathbf{i}}$  can be expressed in terms of the crystal operations in the following way.

**Proposition 5.25** *Let  $Z \in \mathfrak{B}$ . Set  $Z' = (\sigma_{i_1}^* \cdots \sigma_{i_{\ell}}^* \sigma_{i_{\ell}} \cdots \sigma_{i_1})(Z)$  and  $n_k = \varphi_{i_k}(\sigma_{i_{k-1}} \cdots \sigma_{i_1} Z)$  for  $1 \leq k \leq \ell$ . Then  $\Omega_{\mathbf{i}}(Z) = (n_1, \dots, n_{\ell}, Z')$ .*

*Proof.* Set  $i = i_1$  and  $\mathbf{j} = (i_2, \dots, i_{\ell})$ . Let  $T \in Z$  be a general point and let  $X$  be its torsion submodule with respect to  $(\mathcal{T}_1, \mathcal{F}_1)$ . Since  $n_1$  is the dimension of the  $i_1$ -head of  $T$ , we have  $T/X \cong S_{i_1}^{\oplus n_1}$ . On the other hand,  $X$  is a general point of  $\tilde{f}_i^{\max} Z$  and thus  $\Sigma_i X$  is a general point of  $\sigma_i(Z)$ . The module  $X$  is the top step in the filtration of  $T$  defined by our nested sequence of torsion pairs. The other modules define a filtration of  $X$ , whose image by  $\Sigma_i$  is the filtration on  $\Sigma_i X$  defined by the nested sequence

$$(\mathcal{T}_{\ell}, \mathcal{F}_{\ell}) \preceq \cdots \preceq (\mathcal{T}_2, \mathcal{F}_2)$$

of torsion pairs, because  $\Sigma_i$  and  $\Sigma_i^*$  restrict to equivalences of categories

$$\mathcal{T}^{s_{i_1} s_{i_2} \cdots s_{i_k}} \xrightleftharpoons[\Sigma_i^*]{\Sigma_i} \mathcal{F}_{s_i} \cap \mathcal{T}^{s_{i_2} \cdots s_{i_k}} \quad \text{and} \quad \mathcal{F}^{s_{i_1} s_{i_2} \cdots s_{i_k}} \cap \mathcal{T}^{s_i} \xrightleftharpoons[\Sigma_i^*]{\Sigma_i} \mathcal{F}^{s_{i_2} \cdots s_{i_k}}$$

(Propositions 5.7 and 5.17). In addition, we have by induction  $\Omega_{\mathbf{j}}(\sigma_i Z) = (n_2, \dots, n_{\ell}, Z'')$ , where  $Z'' = \sigma_{i_2}^* \cdots \sigma_{i_{\ell}}^* \sigma_{i_{\ell}} \cdots \sigma_{i_2}(\sigma_i Z)$ . The result now follows from Proposition 5.24 (ii).  $\square$

*Remarks 5.26.* (i) Obviously,  $\Omega_{\mathbf{i}}$  induces a bijection between  $\mathfrak{F}^w$  and  $\mathbb{N}^l$ . Up to duality, this bijection corresponds to the parametrization defined by Geiß, Leclerc and Schröer ([24], Proposition 14.5), as one sees from our discussion in Example 5.15.

- (ii) Consider the case where  $\mathfrak{g}$  is finite dimensional and  $w = w_0$ . Now  $\Omega_{\mathbf{i}}$  induces a bijection between  $\mathfrak{B}$  and  $\mathbb{N}^l$ . For  $b \in B(-\infty)$ , the procedure to compute  $\Omega_{\mathbf{i}}(\Lambda_b)$  given in Proposition 5.25 coincides with Saito's method to determine the Lusztig datum of  $b$  in direction  $\mathbf{i}$  (compare with the proof of [50], Lemma 4.1.3). So in this case,  $\Omega_{\mathbf{i}}$  gives the usual Lusztig data.\* This gives us an incentive to use  $\Omega_{\mathbf{i}}$  in order to define Lusztig data in the affine type case as well; we will pursue this road in Section 7.7.

We now examine how  $\Omega_{\mathbf{i}}$  depends on  $\mathbf{i}$ , for a fixed  $w$ . Matsumoto's lemma tells us that the first step is to understand what happens under a braid move, so let us locate a subword of the form  $(i, j, i)$  in  $\mathbf{i}$ , where  $i$  and  $j$  are linked with a single edge in the graph  $(I, E)$ . Let us denote by  $m + 1$  the index at which this subword begins and let us denote by  $\mathbf{j}$  the result of the substitution of  $(i, j, i)$  by  $(j, i, j)$  in  $\mathbf{i}$ .

**Proposition 5.27** *Let  $Z \in \mathfrak{B}$  and write  $\Omega^{\mathbf{i}}(Z) = (n_1, \dots, n_m, p, q, r, n_{m+4}, \dots, n_\ell, Z')$ . Define*

$$q' = \min(p, r), \quad p' + q' = q + r, \quad q' + r' = p + q.$$

*Then  $\Omega^{\mathbf{j}}(Z) = (n_1, \dots, n_m, p', q', r', n_{m+4}, \dots, n_\ell, Z')$ .*

*Proof.* We adopt the notation of the statement of the proposition. Let us set  $J = \{i, j\}$  and  $w = w_J$ .

The nested sequences of torsion pairs defined by  $\mathbf{i}$  and  $\mathbf{j}$  differ only in the places  $m + 1$ ,  $m + 2$  and  $m + 3$ , so  $\Omega_{\mathbf{i}}(Z)$  and  $\Omega_{\mathbf{j}}(Z)$  differ only there. Moreover, by Proposition 5.25, we can assume that  $m = 0$ , by replacing  $Z$  by  $\sigma_{i_m} \cdots \sigma_{i_1}(Z)$ . Finally, we can also assume without loss of generality that  $\ell = 3$  (whence  $w = w_J$ ) and that  $Z \in \mathfrak{F}^{w_J}$ .

By Example 5.6 (ii), the category  $\mathcal{F}^{w_J}$  is isomorphic to the category of representations of the preprojective algebra  $\Lambda_J$ , of type  $A_2$ . As is well-known, this category has four indecomposable objects; further, the irreducible components of the corresponding nilpotent varieties  $\Lambda_J(\nu)$  are rigid and can be described explicitly.

Our  $Z$  is precisely such a component. There is thus  $(a, b, c, d) \in \mathbb{N}^4$  with  $\min(a, b) = 0$  such that a general point in  $Z$  is isomorphic to

$$S_i^{\oplus a} \oplus S_j^{\oplus b} \oplus \left( \begin{array}{c} i \\ \searrow \\ j \end{array} \right)^{\oplus c} \oplus \left( \begin{array}{c} j \\ \swarrow \\ i \end{array} \right)^{\oplus d}.$$

---

\*This result has also been very recently obtained by Yong Jiang. His preprint (*Parametrizations of canonical bases and irreducible components of nilpotent varieties*, arXiv:1110.2937) was posted on the arXiv just when the authors were putting the final touches to this article.



One then easily computes

$$\Omega^{(i,j,i)}(Z) = (a + c, d, b + c, \{0\}) \quad \text{and} \quad \Omega^{(j,i,j)}(Z) = (b + d, c, a + d, \{0\}).$$

Setting  $p = a + c$ ,  $q = d$  and  $r = b + c$ , one checks that  $p' = b + d$ ,  $q' = c$ ,  $r' = a + d$ , showing the desired result.  $\square$

Likewise, let us locate a subword of the form  $(i, j)$  in  $\mathbf{i}$ , where  $i$  and  $j$  are not linked in the graph  $(I, E)$ . Let us denote by  $m + 1$  the index at which this subword begins and let us denote by  $\mathbf{j}$  the result of the substitution of  $(i, j)$  by  $(j, i)$  in  $\mathbf{i}$ .

**Proposition 5.28** *Let  $Z \in \mathfrak{B}$  and write  $\Omega^{\mathbf{i}}(Z) = (n_1, \dots, n_m, p, q, n_{m+3}, \dots, n_\ell, Z')$ . Then  $\Omega^{\mathbf{j}}(Z) = (n_1, \dots, n_m, q, p, n_{m+3}, \dots, n_\ell, Z')$ .*

The proof is similar to that of Proposition 5.27.

For a fixed  $w \in W$ , let  $\mathcal{X}(w)$  be the set of all tuples  $\mathbf{i}$  that represent a reduced decomposition of  $w$ . Lusztig's piecewise linear bijections  $R_{\mathbf{i}}^{\mathbf{j}} : \mathbb{N}^\ell \rightarrow \mathbb{N}^\ell$  ([37], Section 2.1) can be defined here just as in the case where  $w$  is the longest element in a finite  $W$ . Since any two elements in  $\mathcal{X}(w)$  can be related by a sequence of braid and of commutation relations, Propositions 5.27 and 5.28 say that the numerical parts of  $\Omega_{\mathbf{i}}$  and  $\Omega_{\mathbf{j}}$  are related by  $R_{\mathbf{i}}^{\mathbf{j}}$ .

To conclude, let us consider again Proposition 5.25 and choose a general point  $T \in Z$ . The integers  $n_1, \dots, n_\ell$  are equal to the length of the edges of  $\text{Pol}(T)$  along the path

$$\underline{\dim} T, \quad \underline{\dim} T^{s_{i_1}}, \quad \underline{\dim} T^{s_{i_1}s_{i_2}}, \quad \dots, \quad \underline{\dim} T^{s_{i_1}\dots s_{i_\ell}}.$$

Now look at Proposition 5.27, choose  $\theta \in F_J$  and set  $u = s_{i_1} \dots s_{i_m}$ ; then  $u$  is  $J$ -reduced on the right. Using Theorem 5.19 (i) and Proposition 5.24, one can show that the vertices on the face of  $\text{Pol}(T)$  defined by  $u\theta$  are the six vertices  $\underline{\dim} T^{uv}$ , with  $v$  in the parabolic subgroup  $W_J$ . The relation given by Proposition 5.27 constrains the length of the edges of this face; it is equivalent to the tropical Plücker relations of [32].

## 5.6 The finite type case

Our main focus of interest in this paper concerns the case where  $\Phi$  is of affine type, to which Section 5.2 directly applies. In the finite type case, the ideals  $I_w$  are not tilting of projective

dimension at most 1 anymore. They nevertheless exist, so we can define the full subcategories

$$\begin{aligned}\mathcal{T}^w &= \text{essential image of } I_w \otimes_{\Lambda} ?, \\ \mathcal{F}^w &= \text{kernel of } \text{Hom}_{\Lambda}(I_w, ?), \\ \mathcal{T}_w &= \text{kernel of } I_w \otimes_{\Lambda} ?, \\ \mathcal{F}_w &= \text{essential image of } \text{Hom}_{\Lambda}(I_w, ?)\end{aligned}$$

of  $\Lambda\text{-mod}$ . With this definition, all the results in Sections 5.1 hold unchanged, except of course Theorem 5.3 (iii).

To show this, one can adopt the method of Iyama, Reiten and their collaborators, namely, one chooses an embedding of the Dynkin diagram into a non-Dynkin one. One then get a natural surjective morphism from the preprojective algebra  $\widehat{\Lambda}$  of non-Dynkin type onto the preprojective algebra  $\Lambda$  of Dynkin type, and thus a natural embedding of  $\Lambda\text{-mod}$  as a full subcategory of  $\widehat{\Lambda}\text{-mod}$ . This subcategory is abelian and closed under extensions. Each  $i \in I$  yields then an ideal  $I_i$  of  $\Lambda$  and an ideal  $\widehat{I}_i$  of  $\widehat{\Lambda}$ . By Proposition 5.1, the functors  $I_i \otimes_{\Lambda} ?$  and  $\widehat{I}_i \otimes_{\widehat{\Lambda}} ?$  (respectively,  $\text{Hom}_{\Lambda}(I_i, ?)$  and  $\text{Hom}_{\widehat{\Lambda}}(\widehat{I}_i, ?)$ ) coincide on  $\Lambda\text{-mod}$ .

Moreover, the Weyl group  $W$  of the Dynkin diagram embeds as a parabolic subgroup of the Weyl group  $\widehat{W}$  of the larger diagram. Given  $w \in W$ , one can then define  $I_w$  as the bimodule  $I_{i_1} \otimes_{\Lambda} \cdots \otimes_{\Lambda} I_{i_\ell}$  by choosing a reduced decomposition  $w = s_{i_1} \cdots s_{i_\ell}$ . Since the functors  $I_w \otimes_{\Lambda} ?$  and  $\widehat{I}_w \otimes_{\widehat{\Lambda}} ?$  coincide on  $\Lambda\text{-mod}$ , we deduce the Dynkin case of Theorem 5.3 (iv) from the non-Dynkin case. The proof of [13], Proposition II.1.5 then immediately implies Theorem 5.3 (i). Using that moreover the functors  $\text{Hom}_{\Lambda}(I_w, ?)$  and  $\text{Hom}_{\widehat{\Lambda}}(\widehat{I}_w, ?)$  coincide on  $\Lambda\text{-mod}$ , we deduce the Dynkin case from the non-Dynkin case in Theorem 5.4, Examples 5.6 (i) and (ii), Propositions 5.7 and Corollary 5.8.

Finally, to see that Examples 5.6 (iii) and (iv) hold true in the Dynkin case, one can apply Example 5.14. One may here moreover note that the surjection  $\widehat{\Lambda} \rightarrow \Lambda$  induces an isomorphism  $\widehat{\Lambda}/\widehat{I}_w \cong \Lambda/I_w$  (this follows, for instance, by an obvious dimension argument based on Amiot, Iyama, Reiten and Todorov's filtration, see Section 2 of [1]).

Another peculiarity of the finite type case is the fact that the Tits cone  $C_T$  fills us the whole dual space of  $\mathbb{R}I$ . By Theorem 5.19, any torsion pair  $(\overline{\mathcal{T}}_{\theta}, \overline{\mathcal{P}}_{\theta})$  or  $(\mathcal{I}_{\theta}, \overline{\mathcal{P}}_{\theta})$  is of the form  $(\mathcal{T}_w, \mathcal{F}_w)$ , so these latter are enough to completely describe the polytopes  $\text{Pol}(T)$ . This fact, combined with Corollary 5.21, shows that the definition of  $\text{Pol}(T)$  given in [5] is identical to the definition used in the present paper.

In addition, the intersection between  $C_T$  and its opposite is not  $\{0\}$  anymore, so it is possible to write  $-u^{-1}\eta = v\theta$  with  $(u, v) \in W^2$  and both  $\eta$  and  $\theta$  in  $C_0$ . In this case, we have

$$(\mathcal{T}^v, \mathcal{F}^v) = (\mathcal{I}_{v\theta}, \mathcal{I}_{v\theta}) = (\mathcal{I}_{-u^{-1}\eta}, \mathcal{P}_{-u^{-1}\eta}) = (\mathcal{T}_u, \mathcal{F}_u), \quad (5.4)$$

by Theorem 5.19 (ii) and Remark 5.20 (ii).

A last peculiarity of the finite type case was mentioned above in Remark 5.26 (ii).

## 6 The Hall functors

In this section, assume we are given two orthogonal rigid bricks  $S$  and  $R$  in  $\Lambda\text{-mod}$  such that  $\dim \text{Ext}^1(S, R) = 2$ . In other words, we assume that

$$\begin{aligned} \text{End}_\Lambda(S) = \text{End}_\Lambda(R) = K, & \quad \text{Hom}_\Lambda(S, R) = \text{Hom}_\Lambda(R, S) = 0, \\ \text{Ext}_\Lambda^1(S, S) = \text{Ext}_\Lambda^1(R, R) = 0, & \quad \dim \text{Ext}_\Lambda^1(S, R) = 2. \end{aligned} \quad (6.1)$$

We fix  $\xi$  and  $\eta$  in  $\bigoplus_{a \in H} \text{Hom}_K(S_{s(a)}, R_{t(a)})$  and  $\xi^*$  and  $\eta^*$  in  $\bigoplus_{a \in H} \text{Hom}_K(R_{s(a)}, S_{t(a)})$  such that, in the notation of Section 4.2,

$$\begin{aligned} d_{S,R}^1(\xi) = d_{S,R}^1(\eta) = 0, & \quad d_{R,S}^1(\xi^*) = d_{R,S}^1(\eta^*) = 0, \\ \tau_1(\xi, \xi^*) = \tau_1(\eta, \eta^*) = 1, & \quad \tau_1(\xi, \eta^*) = \tau_1(\eta, \xi^*) = 0. \end{aligned} \quad (6.2)$$

Thus  $(\xi, \eta)$  can be regarded as a basis of  $\text{Ext}_\Lambda^1(S, R)$  and  $(\xi^*, \eta^*)$  can be regarded as the dual basis of  $\text{Ext}_\Lambda^1(R, S)$ .

### 6.1 A combinatorial lemma

We denote by  $\Pi$  the completed preprojective algebra of type  $\tilde{A}_1$ . This is the completed preprojective algebra of the Kronecker quiver

$$0 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 1.$$

It contains orthogonal idempotents  $e_0$  and  $e_1$  and arrows  $\alpha, \beta \in e_1 \Pi e_0$  and  $\alpha^*, \beta^* \in e_0 \Pi e_1$ . We denote its augmentation ideal by  $I$ .

For the following, it is useful to think of the algebra  $\Pi$  as the quotient

$$\mathbf{S} \langle \langle \alpha, \beta, \alpha^*, \beta^* \rangle \rangle / (\alpha \alpha^* + \beta \beta^*, \alpha^* \alpha + \beta^* \beta)$$

of the ring of non-commutative formal power series in four variables  $\alpha, \beta, \alpha^*, \beta^*$  with coefficients in the commutative semisimple algebra  $\mathbf{S} = Ke_0 \oplus Ke_1$ .

**Lemma 6.1** *The image of the linear map*

$$C : (e_1 \Pi e_0)^2 \times (e_0 \Pi e_1)^2 \rightarrow e_0 \Pi e_0 \times e_1 \Pi e_1$$

*given by*

$$C(x, y, x^*, y^*) = (-x^* \alpha - y^* \beta - \alpha^* x - \beta^* y, x \alpha^* + y \beta^* + \alpha x^* + \beta y^*)$$

*contains:*

- any element of the form  $(uv - vu, 0)$ , where  $(u, v) \in (e_0 \Pi e_0)^2$ ;
- any element of the form  $(0, uv - vu)$ , where  $(u, v) \in (e_1 \Pi e_1)^2$ ;
- any element of the form  $(uv, -vu)$ , where  $(u, v) \in e_0 \Pi e_1 \times e_1 \Pi e_0$ .

*Proof.* Let  $u$  and  $v$  be two words of even length in the alphabet  $\{\alpha, \beta, \alpha^*, \beta^*\}$ , in which starred and non-starred letters alternate, and which start with a starred letter. Thus  $u$  and  $v$  define elements in  $e_0 \Pi e_0$ . We write  $u = c_1^* c_2 \cdots c_{2\ell-1}^* c_{2\ell}$ , with  $c_k \in \{\alpha, \beta\}$ . For  $1 \leq k \leq \ell$ , we set

$$m_k = c_{2k} c_{2k+1}^* \cdots c_{2\ell} v c_1^* c_2 \cdots c_{2k-2} \quad \text{and} \quad n_k = c_{2k+1}^* c_{2k+2} \cdots c_{2\ell} v c_1^* c_2 \cdots c_{2k-1}^*.$$

Then  $(uv - vu, 0)$  is the image of the element

$$\left( - \sum_{\substack{1 \leq k \leq \ell \\ c_{2k-1}^* = \alpha^*}} m_k, - \sum_{\substack{1 \leq k \leq \ell \\ c_{2k-1}^* = \beta^*}} m_k, \sum_{\substack{1 \leq k \leq \ell \\ c_{2k} = \alpha}} n_k, \sum_{\substack{1 \leq k \leq \ell \\ c_{2k} = \beta}} n_k \right)$$

by our linear map. This shows that the elements of the first kind belong to the image of our map. The two other cases are similar.  $\square$

## 6.2 A universal lifting

We set  $P = \xi \otimes \alpha + \eta \otimes \beta$  and  $Q = \xi^* \otimes \alpha^* + \eta^* \otimes \beta^*$ ; these are elements in

$$\bigoplus_{a \in H} \text{Hom}_K(S_{s(a)}, R_{t(a)}) \otimes_K e_1 \Pi e_0 \quad \text{and} \quad \bigoplus_{a \in H} \text{Hom}_K(R_{s(a)}, S_{t(a)}) \otimes_K e_0 \Pi e_1,$$

respectively.

**Lemma 6.2** *There are elements*

$$\begin{aligned} S^{(\infty)} &\in \bigoplus_{a \in H} \text{Hom}_K(S_{s(a)}, S_{t(a)}) \otimes_K e_0 \Pi e_0, & R^{(\infty)} &\in \bigoplus_{a \in H} \text{Hom}_K(R_{s(a)}, R_{t(a)}) \otimes_K e_1 \Pi e_1, \\ P^{(\infty)} &\in \bigoplus_{a \in H} \text{Hom}_K(S_{s(a)}, R_{t(a)}) \otimes_K e_1 \Pi e_0, & Q^{(\infty)} &\in \bigoplus_{a \in H} \text{Hom}_K(R_{s(a)}, S_{t(a)}) \otimes_K e_0 \Pi e_1 \end{aligned}$$

such that for each  $a \in H$ ,

$$\begin{pmatrix} S_a^{(\infty)} & Q_a^{(\infty)} \\ P_a^{(\infty)} & R_a^{(\infty)} \end{pmatrix} \equiv \begin{pmatrix} S_a & Q_a \\ P_a & R_a \end{pmatrix} \pmod{\begin{pmatrix} I^2 & I^3 \\ I^3 & I^2 \end{pmatrix}},$$

and for each  $i \in I$ ,

$$\sum_{\substack{a \in H \\ s(a)=i}} \varepsilon(a) \begin{pmatrix} S_{a^*}^{(\infty)} & Q_{a^*}^{(\infty)} \\ P_{a^*}^{(\infty)} & R_{a^*}^{(\infty)} \end{pmatrix} \begin{pmatrix} S_a^{(\infty)} & Q_a^{(\infty)} \\ P_a^{(\infty)} & R_a^{(\infty)} \end{pmatrix} = 0.$$

*Proof.* The desired elements will be constructed as the limit in the  $I$ -adic topology of elements  $S^{(k)}$ ,  $R^{(k)}$ ,  $P^{(k)}$  and  $Q^{(k)}$  such that

$$\begin{pmatrix} S^{(k+1)} & Q^{(k+1)} \\ P^{(k+1)} & R^{(k+1)} \end{pmatrix} \equiv \begin{pmatrix} S^{(k)} & Q^{(k)} \\ P^{(k)} & R^{(k)} \end{pmatrix} \pmod{\begin{pmatrix} I^{2k+2} & I^{2k+3} \\ I^{2k+3} & I^{2k+2} \end{pmatrix}}.$$

For  $k = 0$ , we define

$$S^{(0)} \in \bigoplus_{a \in H} \text{Hom}_K(S_{s(a)}, S_{t(a)})$$

by gathering the structure maps  $S_a$  of the  $\Lambda$ -module  $S$ . We similarly define  $R^{(0)}$  and we set  $P^{(0)} = P$  and  $Q^{(0)} = Q$ . The conditions we impose at step  $k$  are

$$\sum_{\substack{a \in H \\ s(a)=i}} \varepsilon(a) \begin{pmatrix} S_{a^*}^{(k)} & Q_{a^*}^{(k)} \\ P_{a^*}^{(k)} & R_{a^*}^{(k)} \end{pmatrix} \begin{pmatrix} S_a^{(k)} & Q_a^{(k)} \\ P_a^{(k)} & R_a^{(k)} \end{pmatrix} \equiv 0 \pmod{\begin{pmatrix} I^{2k+2} & I^{2k+3} \\ I^{2k+3} & I^{2k+2} \end{pmatrix}} \quad (6.4)$$

for each  $i \in I$ , and in the notation of Section 4.2

$$\tau_1(S^{(k)}, S^{(k)}) + \tau_1(Q^{(k)}, P^{(k)}) \equiv 0 \pmod{I^{2k+4}} \quad (6.5)$$

and

$$\tau_1(R^{(k)}, R^{(k)}) + \tau_1(P^{(k)}, Q^{(k)}) \equiv 0 \pmod{I^{2k+4}}. \quad (6.6)$$

Thanks to the preprojective relations in  $S$  and  $R$  and to Equations (6.2) and (6.3), these conditions are fulfilled at step  $k = 0$ .

Assume that  $S^{(k)}$ ,  $R^{(k)}$ ,  $P^{(k)}$  and  $Q^{(k)}$  have been constructed. We set

$$f_i = \sum_{\substack{a \in H \\ s(a)=i}} \varepsilon(a) \left( S_{a^*}^{(k)} S_a^{(k)} + Q_{a^*}^{(k)} P_a^{(k)} \right)$$

and regard it as a formal series in  $\alpha$ ,  $\beta$ ,  $\alpha^*$ ,  $\beta^*$  with coefficients in  $\bigoplus_{i \in I} \text{Hom}_K(S_i, S_i)$  and valuation at least  $2k + 2$ . Then, thanks to (6.5), we have

$$\tau_2((f_i), (\text{id}_{S_i})) \in I^{2k+4};$$

in other words, the coefficient in  $(f_i)$  of any monomial of degree less than  $2k + 4$  is  $\tau_2$ -orthogonal to  $\text{Hom}_K(S, S)$ . We conclude that modulo  $I^{2k+4}$ ,  $(f_i)$  belongs to  $d_{M,M}^1$ . Therefore there exists

$$\tilde{S} \in \bigoplus_{a \in H} \text{Hom}_K(S_{s(a)}, S_{t(a)}) \otimes_K e_0 \Pi e_0$$

of valuation at least  $2k+2$  such that  $d_{M,M}^1(\tilde{S}) \equiv (f_i) \pmod{I^{2k+4}}$ . We then set  $S^{(k+1)} = S^{(k)} - \tilde{S}$ , and the upper left corner of (6.4) is satisfied at step  $k + 1$ . One similarly finds  $R^{(k+1)}$ ,  $P^{(k+1)}$  and  $Q^{(k+1)}$  such that (6.4) is satisfied at step  $k + 1$ .

Now let

$$D = \left( \tau_1(S^{(k+1)}, S^{(k+1)}) + \tau_1(Q^{(k+1)}, P^{(k+1)}), \tau_1(R^{(k+1)}, R^{(k+1)}) + \tau_1(P^{(k+1)}, Q^{(k+1)}) \right).$$

Thus  $D \in e_0 \Pi e_0 \times e_1 \Pi e_1$  and his two components have valuations at least  $2k + 4$ . Because of the cyclicity of the trace and of the presence of the sign  $\varepsilon(a)$  in

$$\tau_1(S^{(k+1)}, S^{(k+1)}) = \sum_{i \in I} \text{Tr} \left( \sum_{\substack{a \in H \\ s(a)=i}} \varepsilon(a) \left( S_{a^*}^{(k)} S_a^{(k)} \right) \right),$$

the first term in the first component of  $D$  is a linear combination of elements of the kind  $uv - vu$ , with  $(u, v) \in (e_0 \Pi e_0)^2$ . Likewise, we see that the contributions to  $D$  of  $R^{(k+1)}$ ,  $P^{(k+1)}$  and  $Q^{(k+1)}$  are linear combination elements of the second and third kind in the statement of Lemma 6.1. Therefore  $D$  can be written as  $C(x, y, x^*, y^*)$ , with the notation of that lemma, where  $(x, y) \in (e_1 \Pi e_0)^2$  and  $(x^*, y^*) \in (e_0 \Pi e_1)^2$  are of valuation at least  $2k + 3$ . If we now subtract  $\xi \otimes x + \eta \otimes y$  and  $\xi^* \otimes x^* + \eta^* \otimes y^*$  from  $P^{(k+1)}$  and  $Q^{(k+1)}$ , respectively,  $D$  will vanish modulo  $I^{2k+6}$ . Moreover, the condition (6.4) will still be satisfied, thanks to (6.2).  $\square$

### 6.3 Construction of the Hall functors

We are now in a position to define a full and faithful exact functor  $\mathcal{H} : \Pi\text{-mod} \rightarrow \Lambda\text{-mod}$ .

Let  $V$  be a  $\Pi$ -module. We define an  $I$ -graded vector space  $M$  by

$$M_i = (S_i \otimes_K V_0) \oplus (R_i \otimes_K V_1).$$

Now let  $a \in H$ . Then  $S_a^{(\infty)}$  is an element of  $\text{Hom}_K(S_{s(a)}, S_{t(a)}) \otimes_K e_0 \Pi e_0$ , hence can be seen as a matrix valued element in  $e_0 \Pi e_0$ . We can evaluate the structure maps of  $V$  at this element; the result belongs to

$$\text{Hom}_K(S_{s(a)}, S_{t(a)}) \otimes_K \text{Hom}_K(V_0, V_0) = \text{Hom}_K(S_{s(a)} \otimes_K V_0, S_{t(a)} \otimes_K V_0).$$

Evaluating  $R_a^{(\infty)}$ ,  $P_a^{(\infty)}$  and  $Q_a^{(\infty)}$  in a similar way, we map

$$\begin{pmatrix} S_a^{(\infty)} & Q_a^{(\infty)} \\ P_a^{(\infty)} & R_a^{(\infty)} \end{pmatrix}$$

to an element in  $\text{Hom}_K(M_{s(a)}, M_{t(a)})$ . The conditions imposed in Lemma 6.2 assert that we then get a  $\Lambda$ -module  $M$ .

Since we have worked with a universal formula (the same for all  $\Pi$ -modules  $V$ ), the assignment  $V \mapsto M$  defines a functor  $\mathcal{H}$ .

This functor  $\mathcal{H}$  maps the two simple  $\Pi$ -modules to  $S$  and  $R$ . Moreover, it is exact. Let  $\langle S, R \rangle$  be the smallest abelian, closed under extensions, subcategory of  $\Lambda\text{-mod}$  that contains (the isomorphism classes of)  $S$  and  $R$ .

**Theorem 6.3** *The functor  $\mathcal{H}$  induces an equivalence of categories between  $\Pi\text{-mod}$  and  $\langle S, R \rangle$ .*

*Proof.* The category  $\langle S, R \rangle$  has only (up to isomorphism) two simple objects, namely  $S$  and  $R$ , for these latter are orthogonal bricks. In view of [20], Lemma 11.7, it thus suffices to show that for any simple  $\Pi$ -modules  $L$  and  $L'$ , the induced homomorphism  $\text{Ext}_{\Pi}^k(L, L') \rightarrow \text{Ext}_{\langle S, R \rangle}^k(\mathcal{H}(L), \mathcal{H}(L'))$  is bijective for  $k \in \{0, 1\}$  and injective for  $k = 2$ . We can here replace the extension spaces in  $\langle S, R \rangle$  by the extension spaces in  $\Lambda\text{-mod}$ : this does not change the  $\text{Ext}^0$  nor the  $\text{Ext}^1$ , for  $\langle S, R \rangle$  is full and closed under extensions; and if the injectivity condition holds for  $\text{Ext}_{\Lambda}^2$ , it will a fortiori holds for  $\text{Ext}_{\langle S, R \rangle}^2$ .

Let us call  $W_0$  and  $W_1$  the two simple  $\Pi$ -modules; then  $\mathcal{H}(W_0) = S$  and  $\mathcal{H}(W_1) = R$ . Obviously,

$$\text{End}(W_0) = \text{End}(W_1) = K \quad \text{and} \quad \text{Hom}(W_0, W_1) = \text{Hom}(W_1, W_0) = 0,$$

so the condition is fulfilled for  $k = 0$ .

The  $\Pi$ -modules  $T_\alpha$  and  $T_\beta$  with dimension-vector  $(1, 1)$  obtained by letting the arrows of  $\Pi$  act by

$$(\alpha, \beta, \alpha^*, \beta^*) \mapsto (1, 0, 0, 0) \quad \text{and} \quad (\alpha, \beta, \alpha^*, \beta^*) \mapsto (0, 1, 0, 0)$$

are extensions of  $W_0$  by  $W_1$ . We denote their extension classes in  $\text{Ext}_\Pi^1(W_0, W_1)$  by  $\alpha$  and  $\beta$ , respectively. The extension classes of  $\mathcal{H}(T_\alpha)$  and  $\mathcal{H}(T_\beta)$  are  $\xi$  and  $\eta$ . Thus, the induced homomorphism  $\text{Ext}_\Pi^1(W_0, W_1) \rightarrow \text{Ext}_\Lambda^1(S, R)$  maps the basis  $(\alpha, \beta)$  of the first space to the basis  $(\xi, \eta)$  of the second space; it is therefore bijective. We check in a similar way the other cases for  $k = 1$ .

The equality  $\tau_1(\xi, \xi^*) = 1$  implies that the Yoneda product  $\xi\xi^* \in \text{Ext}_\Lambda^2(R, R)$  does not vanish. The induced homomorphism  $\text{Ext}_\Pi^2(W_1, W_1) \rightarrow \text{Ext}_\Lambda^2(R, R)$  maps  $\alpha\alpha^*$  to  $\xi\xi^*$ , so it cannot be zero. It is thus injective, for  $\text{Ext}_\Pi^2(W_1, W_1)$  is one dimensional. The other cases for  $k = 2$  are treated in like manner.  $\square$

We here note that a proof for Lemma 11.7 in [20] can be found in [35], Proposition 3.4.3.

## 6.4 Irreducible components

We now study the consequences of the existence of a Hall functor at the level of irreducible components of the nilpotent varieties.

Let  $\mu = (\mu_0, \mu_1)$  be a dimension-vector for  $\Pi$  and set  $\nu = \mu_0 \underline{\dim} S + \mu_1 \underline{\dim} R$ . We denote by  $\Pi(\mu)$  the nilpotent variety for  $\Pi$  and by  $\Lambda_{\langle S, R \rangle}(\nu)$  the set of all points in  $\Lambda(\nu)$  that belong to  $\langle S, R \rangle$ . In addition, we define  $\Omega(\mu)$  to be the set of all triples  $(V, M, f)$  such that  $V \in \Pi(\mu)$ ,  $M \in \Lambda(\nu)$  and  $f : \mathcal{H}(V) \rightarrow M$  is an isomorphism of  $\Lambda$ -modules. We can then form the diagram

$$\Pi(\mu) \xleftarrow{p} \Omega(\mu) \xrightarrow{q} \Lambda(\nu) \tag{6.7}$$

in which  $p$  and  $q$  are the first and second projection. Obviously,  $p$  is a principal  $G(\nu)$ -bundle, the image of  $q$  is  $\Lambda_{\langle S, R \rangle}(\nu)$ , and each non-empty fiber of  $q$  is isomorphic to  $G(\mu)$ .

**Proposition 6.4** *The subset  $\Lambda_{\langle S, R \rangle}(\nu)$  is constructible and all its irreducible components have full dimension in  $\Lambda(\nu)$ . The diagram (6.7) induces a bijection between the irreducible components of  $\Pi(\mu)$  and the irreducible components of  $\Lambda(\nu)$  whose general point belongs to  $\Lambda_{\langle S, R \rangle}(\nu)$ .*

*Proof.* Combining the conditions (6.1) with equation (4.2), we get

$$(\mu, \mu) = (\nu, \nu),$$



where  $(, )$  in the left-hand side is the bilinear form on  $K_0(\Pi\text{-mod})$  and  $(, )$  in the right-hand side is the bilinear form on  $K_0(\Lambda\text{-mod}) = \mathbb{Z}I$ . In view of (4.5), this translates to

$$\dim G(\mu) - \dim \Pi(\mu) = \dim G(\nu) - \dim \Lambda(\nu).$$

The proposition now results from standard arguments of algebraic geometry, similar to those used in Section 4.5.  $\square$

## 7 Preprojective algebras of affine type

From now on,  $\mathfrak{g}$  is of symmetric affine type. We will apply the theory we have been developing to finally obtain our affine MV polytopes. We will use the notation concerning affine roots systems discussed in Section 2.2.

### 7.1 Vertical faces

Given a  $\Lambda$ -module  $T$ , we have already seen that the support function of  $\text{Pol}(T)$  is linear on each cone  $\pm wF_J$  (Corollary 5.22). To complete the picture, we now show that it is also linear on each face  $F$  of the spherical Weyl fan.

Recall that the Weyl group  $W$  contains as a normal subgroup the set of all translations  $t_\lambda$ , for  $\lambda$  in the coroot lattice  $Q^\vee \subseteq \mathfrak{t}_{\mathbb{R}}$  of  $\Phi^s$ . The translation  $t_\lambda$  acts on  $\mathbb{R}I$  by  $t_\lambda \nu = \nu - \langle \lambda, \nu \rangle \delta$ .

**Proposition 7.1** *Let  $\lambda \in Q^\vee$ . Then*

$$\begin{aligned} \mathcal{I}_\lambda &= \bigcup_{n \in \mathbb{N}} \mathcal{I}_{t_{n\lambda}}, & \overline{\mathcal{I}}_\lambda &= \bigcap_{n \in \mathbb{N}} \mathcal{I}^{t_{n\lambda}}, \\ \mathcal{P}_\lambda &= \bigcup_{n \in \mathbb{N}} \mathcal{P}^{t_{n\lambda}}, & \overline{\mathcal{P}}_\lambda &= \bigcap_{n \in \mathbb{N}} \mathcal{P}_{t_{n\lambda}}, \\ \text{and } \mathcal{R}_\lambda &= \bigcap_{n \in \mathbb{N}} (\mathcal{F}_{t_{n\lambda}} \cap \mathcal{I}^{t_{n\lambda}}). \end{aligned}$$

*Proof.* We first note that by (5.3), the unions here are nondecreasing with  $n$  and the intersection is nonincreasing. Let us pick  $\theta \in C_0$  and let  $T$  be a  $\Lambda$ -module.

Suppose that  $T$  belongs to  $\mathcal{I}_{t_{n\lambda}}$  for some  $n$ , and let  $X$  be a nonzero quotient of  $T$ . By Remark 5.20 (ii),  $T \in \mathcal{I}_{-t_{-n\lambda}\theta}$ , hence  $\langle -t_{-n\lambda}\theta, \underline{\dim} X \rangle > 0$ . Here the left-hand side is

$\langle -\theta, t_{n\lambda} \underline{\dim} X \rangle = -\langle \theta, \underline{\dim} X \rangle + n\langle \lambda, \underline{\dim} X \rangle \langle \theta, \delta \rangle$ , so we necessarily have  $\langle \lambda, \underline{\dim} X \rangle > 0$ . We conclude that  $T \in \mathcal{I}_\lambda$ .

Conversely, suppose that  $T \in \mathcal{I}_\lambda$  and let  $X$  be a nonzero quotient of  $T$ . Then  $\langle \lambda, \underline{\dim} X \rangle > 0$ , and thus  $\langle -t_{-n\lambda}\theta, \underline{\dim} X \rangle > 0$  for  $n$  large enough. Since there is only a finite number of possibilities for  $\underline{\dim} X$ , we can choose  $n$  independently of  $X$ . Therefore  $T$  belongs to  $\mathcal{T}_{t_{n\lambda}} = \mathcal{I}_{-t_{-n\lambda}\theta}$ .

We have thus obtained the first equality. Similar reasonings establish the three next ones. The last equality comes from the general relation  $\mathcal{R}_\lambda = \overline{\mathcal{I}}_\lambda \cap \overline{\mathcal{P}}_\lambda$ .  $\square$

**Corollary 7.2** *Let  $F$  be a face in the spherical Weyl fan. Then  $\mathcal{I}_\lambda$ ,  $\overline{\mathcal{I}}_\lambda$ ,  $\mathcal{P}_\lambda$ ,  $\overline{\mathcal{P}}_\lambda$  and  $\mathcal{R}_\lambda$  are all independent on the choice of  $\lambda \in F \cap Q^\vee$ .*

*Proof.* Let  $\lambda$  and  $\mu$  in  $F$ . The set  $A = \{\alpha \in \Phi^s \mid \langle \lambda, \alpha \rangle > 0\}$  would then be the same with  $\lambda$  replaced by  $\mu$ . Let  $M$  and  $K$  be the maxima of the quotients  $\langle \lambda, \alpha \rangle / \langle \mu, \alpha \rangle$  and  $\langle \mu, \alpha \rangle / \langle \lambda, \alpha \rangle$ , for  $\alpha \in A$ . Then

$$N_{t_{-n\lambda}} \subseteq N_{t_{-nM\mu}} \quad \text{and} \quad N_{t_{-n\mu}} \subseteq N_{t_{-nK\lambda}}$$

for any integer  $n \geq 0$ . From Lemma 2.1 and (5.3), it follows that

$$\mathcal{T}_{t_{n\lambda}} \subseteq \mathcal{T}_{t_{nM\mu}} \quad \text{and} \quad \mathcal{T}_{t_{n\mu}} \subseteq \mathcal{T}_{t_{nK\lambda}}.$$

Proposition 7.1 then implies  $\mathcal{I}_\lambda = \mathcal{I}_\mu$ , as desired. The case of the other categories is dealt with in a similar fashion.  $\square$

By approximation of a (large enough positive multiple of)  $\lambda \in F$  by elements in  $F \cap Q^\vee$ , one can extend the validity of Corollary 7.2 to arbitrary  $\lambda \in F$ . It then fully makes sense to denote the categories  $\mathcal{I}_\lambda$ ,  $\overline{\mathcal{I}}_\lambda$ , etc. by  $\mathcal{I}_F$ ,  $\overline{\mathcal{I}}_F$ , etc. Likewise, we denote by  $\mathfrak{I}_F$ ,  $\mathfrak{R}_F$  and  $\mathfrak{P}_F$  the subsets  $\mathfrak{I}_\lambda$ ,  $\mathfrak{R}_\lambda$  and  $\mathfrak{P}_\lambda$  of  $\mathfrak{B}$ .

Together with Corollary 5.22, the above implies:

**Corollary 7.3** *The support function of the HN polytope of a  $\Lambda$ -module is linear on each face of the affine Weyl fan.*

## 7.2 Torsion pairs associated to biconvex subsets

Counting arguments will play a key role in Section 7.5. Anticipating our future needs, we now determine the cardinality of  $\mathfrak{R}_\theta(\nu)$ , the subset of  $\mathfrak{B}(\nu)$  consisting of components whose general

points belong to  $\mathcal{R}_\theta$ , for any dimension-vector  $\nu \in \mathbb{N}I$  and any  $\theta \in (\mathbb{R}I)^*$ . Using the biconvex subsets of Section 2.3 will prove very convenient.

Each  $w \in W$  gives rise to two biconvex sets, namely  $A_w = N_{w-1}$  and  $A^w = \Phi_+ \setminus N_w$ . We set

$$(\mathcal{T}(A_w), \mathcal{F}(A_w)) = (\mathcal{T}_w, \mathcal{F}_w) \quad \text{and} \quad (\mathcal{T}(A^w), \mathcal{F}(A^w)) = (\mathcal{T}^w, \mathcal{F}^w).$$

One may here note that in the case where  $\Phi_+$  is finite, the two possible definitions for  $(\mathcal{T}(A), \mathcal{F}(A))$  agree because of (5.4). One may also note that if  $\theta$  is chosen such that  $A = \{\alpha \in \Phi_+ \mid \langle \theta, \alpha \rangle > 0\}$ , then  $(\mathcal{T}(A), \mathcal{F}(A)) = (\mathcal{T}_\theta, \mathcal{F}_\theta)$ .

In this fashion, we associate a torsion pair in  $\Lambda\text{-mod}$  to each finite or cofinite biconvex set. Using Lemma 2.1, (5.3) and Proposition 5.16, we deduce the following monotonicity property: if  $A \subseteq B$ , then  $(\mathcal{T}(A), \mathcal{F}(A)) \preceq (\mathcal{T}(B), \mathcal{F}(B))$ . We also note the following interpretation of Example 5.6 (iii): if  $B = \Phi_+ \setminus A$ , then  $(\mathcal{T}(B), \mathcal{F}(B)) = (\mathcal{F}(A)^*, \mathcal{T}(A)^*)$ .

By Lemma 2.6 (i), every biconvex subset  $A$  is either the increasing union of finite biconvex subsets or the decreasing intersection of cofinite biconvex subsets. In the former case, we set

$$\mathcal{T}(A) = \bigcup_{\substack{B \text{ finite biconvex} \\ B \subseteq A}} \mathcal{T}(B) \quad \text{and} \quad \mathcal{F}(A) = \bigcap_{\substack{B \text{ finite biconvex} \\ B \subseteq A}} \mathcal{F}(B). \quad (7.1)$$

Then  $(\mathcal{T}(A), \mathcal{F}(A))$  is a torsion pair. The axiom (T1) is indeed easily verified. To check (T2), we take  $T \in \Lambda\text{-mod}$ . Each finite biconvex subset  $B \subseteq A$  provides a submodule  $T_B \subseteq T$  such that  $(T_B, T/T_B) \in \mathcal{T}(B) \times \mathcal{F}(B)$ . The monotonicity property implies that the map  $B \mapsto T_B$  is non-decreasing, so for dimension reasons, the family  $(T_B)$  has an largest element, say  $T_{B_0}$ . Then for any  $B_1 \supseteq B_0$ , we have  $T_{B_1} = T_{B_0}$ , which shows that  $(T_{B_0}, T/T_{B_0}) \in \mathcal{T}(B_1) \times \mathcal{F}(B_1)$ . We conclude that  $(T_{B_0}, T/T_{B_0}) \in \mathcal{T}(A) \times \mathcal{F}(A)$ , as desired.

*Remark 7.4.* In the context above, fix a dimension-vector  $\nu \in \mathbb{N}I$  and a finite biconvex subset  $B_0 \subseteq A$  that contains  $\{\alpha \in A \mid \text{ht } \alpha \leq \text{ht } \nu\}$ . We claim that any  $T \in \mathcal{T}(A)$  of dimension-vector  $\nu$  already belongs to  $\mathcal{T}(B_0)$ . To see this, take  $T$  as indicated, and choose a finite biconvex subset  $B_1 \subseteq A$  such that  $T \in \mathcal{T}(B_1)$ . At the price of replacing  $B_1$  by a larger subset, we may assume that  $B_1$  contains  $B_0$ . Let us write  $B_0 = A_u$  and  $B_1 = A_{vu}$ , with  $(u, v) \in W^2$  such that  $\ell(vu) = \ell(v) + \ell(u)$ , and choose a reduced decomposition of  $vu$  compatible with this factorization. Then  $T \in \mathcal{T}_{vu}$ , so Theorem 5.12 provides a filtration on  $T$ . The dimension-vectors of the subquotients belong to  $B_1$ , but for dimension reasons, they belong in fact to  $B_0$ : all the nonzero subquotients occur while scanning  $u$ . Therefore  $T = T_u$  belongs to  $\mathcal{T}_u = \mathcal{T}(B_0)$ , as announced.

When  $A$  is the intersection of cofinite biconvex subsets, we switch the role of intersection and union above. We then have a torsion pair  $(\mathcal{T}(A), \mathcal{F}(A))$  associated in a monotonous way to each biconvex subset  $A$ .

By Example 2.4 (iii), any  $\theta \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}I, \mathbb{R})$  defines two biconvex subsets

$$A_{\theta}^{\min} = \{\alpha \in \Phi_+ \mid \langle \theta, \alpha \rangle > 0\} \quad \text{and} \quad A_{\theta}^{\max} = \{\alpha \in \Phi_+ \mid \langle \theta, \alpha \rangle \geq 0\}.$$

**Proposition 7.5** *We have*

$$(\mathcal{T}(A_{\theta}^{\min}), \mathcal{F}(A_{\theta}^{\min})) = (\mathcal{I}_{\theta}, \overline{\mathcal{P}}_{\theta}) \quad \text{and} \quad (\mathcal{T}(A_{\theta}^{\max}), \mathcal{F}(A_{\theta}^{\max})) = (\overline{\mathcal{I}}_{\theta}, \mathcal{P}_{\theta}).$$

*Proof.* The case  $\theta = 0$  is straightforward; let us discard it.

Suppose first that  $\theta \in C_T$ . We can then find  $J \subsetneq I$ ,  $\eta \in F_J$  and  $w \in W$  such that  $\theta = w\eta$  and  $w$  is  $J$ -reduced on the right. We note here that  $W_J$  is finite, whence a longest element  $w_J$  in  $W_J$ . The desired result follows then by Proposition 2.5 and Theorem 5.19.

The case where  $\theta \in (-C_T)$  is analogous (see if necessary Remark 4.1 and Example 5.6 (iii)).

Lastly, the case  $\theta \in \mathfrak{t}$  follows from Proposition 7.1, taking into account the equalities (2.1).  $\square$

**Proposition 7.6** *Let  $A$  and  $B$  be two biconvex subsets and let  $\alpha \in \Phi_+^{\text{re}}$ . Assume that  $B = A \sqcup \{\alpha\}$ . Then there is a rigid indecomposable  $\Lambda$ -module  $L(A, B)$  of dimension-vector  $\alpha$  such that  $\mathcal{F}(A) \cap \mathcal{T}(B) = \text{add } L(A, B)$ .*

*Proof.* The case where  $A$  and  $B$  are both finite follows from Theorem 5.12 (i): if  $A = A_w$  and  $B = A_{s_i w}$ , then  $L(A, B) = \text{Hom}_{\Lambda}(I_w, S_i)$ .

Assume now that  $A$  and  $B$  are infinite and that  $\delta \notin A$ . By Lemma 2.7, we can then find  $A' \subseteq A$  and  $B' \subseteq B$  finite biconvex subsets such that  $B' = A' \sqcup \{\alpha\}$ . Certainly,  $A'$  and  $B'$  are not unique subject to these requirements. We claim however that  $L(A', B')$  does not depend on the choice of  $A'$  and  $B'$ .

To see this, consider  $A''$  and  $B''$  finite biconvex subsets with  $B'' = A'' \sqcup \{\alpha\}$  and  $A'' \subseteq A$ . We want to show that  $L(A', B') \cong L(A'', B'')$ . Without loss of generality, we may assume that  $A'' \supseteq A'$ . We write  $A' = A_u$ ,  $B' = A_{s_i u}$ ,  $A'' = A_{vu}$ ,  $B'' = A_{s_j vu}$  with  $\ell(vu) = \ell(v) + \ell(u)$  and  $\alpha = u^{-1}\alpha_i = (vu)^{-1}\alpha_j$ . Then  $L(A', B') = \text{Hom}_{\Lambda}(I_u, S_i)$  and  $L(A'', B'') = \text{Hom}_{\Lambda}(I_{vu}, S_j) = \text{Hom}_{\Lambda}(I_u, \text{Hom}_{\Lambda}(I_v, S_j))$ . Observing that  $\text{Hom}_{\Lambda}(I_v, S_j)$  has dimension-vector  $v^{-1}\alpha_j = \alpha_i$ , we obtain  $L(A', B') \cong L(A'', B'')$ , as announced.

Certainly,  $L(A', B') \in \mathcal{T}(B)$ . Let  $A_0$  be finite biconvex subset contained in  $A$ . By Lemma 2.7, there is a finite biconvex subset  $A'' \subseteq A$  that contains  $A_0$  and such that  $B'' = A'' \sqcup \{\alpha\}$  is biconvex. Then  $L(A', B') \cong L(A'', B'')$  belongs to  $\mathcal{F}(A'')$ , hence to  $\mathcal{F}(A_0)$ . Since  $A_0$  was arbitrary, we get  $L(A', B') \in \mathcal{F}(A)$ . It follows that  $\text{add } L(A', B') \subseteq \mathcal{F}(A) \cap \mathcal{T}(B)$ .

Conversely, let  $T \in \mathcal{F}(A) \cap \mathcal{T}(B)$ , of dimension-vector say  $\nu$ . By Lemma 2.7, there is a biconvex subset  $A'' \subseteq A$  that contains  $\{\beta \in A \mid \text{ht } \beta \leq \text{ht } \nu\}$  and such that  $B'' = A'' \sqcup \{\alpha\}$  is biconvex. By Remark 7.4,  $T \in \mathcal{T}(B'')$ . Therefore  $T$  belongs to  $\mathcal{F}(A'') \cap \mathcal{T}(B'') = \text{add } L(A'', B'')$ .

Setting  $L(A, B) = L(A', B')$ , we thus have  $\mathcal{F}(A) \cap \mathcal{T}(B) = \text{add } L(A, B)$ , as desired.

It remains to deal with the case where  $\delta \in A$ . When  $A$  and  $B$  are both cofinite, the result follows from Theorem 5.11 (i): if  $A = A^{wsi}$  and  $B = A^w$ , then  $L(A, B) = I_w \otimes_{\Lambda} S_i$ . The general case follows by approximation, in a similar fashion as above.  $\square$

Proposition 4.2(iii) implies that the torsion pair  $(\mathcal{T}(A), \mathcal{F}(A))$  satisfies the openness condition (O) for any biconvex subset  $A$  of the form  $A_{\theta}^{\min}$  or  $A_{\theta}^{\max}$ , so in particular for any  $A$  finite or cofinite. Using Remark 7.4, one easily extends this property to any biconvex  $A$ . We may thus apply the results of Section 4.5: each biconvex subset  $A$  defines subsets  $\mathfrak{T}(A)$  and  $\mathfrak{F}(A)$  of  $\mathfrak{B}$  and we have a bijection

$$\Xi(A) : \mathfrak{T}(A) \times \mathfrak{F}(A) \rightarrow \mathfrak{B}.$$

More generally if  $\mathbf{A} = (A_0, \dots, A_{\ell})$  is a nondecreasing list of biconvex subsets, then we have nested torsion pairs

$$(\mathcal{T}(A_0), \mathcal{F}(A_0)) \preceq \dots \preceq (\mathcal{T}(A_{\ell}), \mathcal{F}(A_{\ell})),$$

whence a bijection

$$\Xi(\mathbf{A}) : \mathfrak{T}(A_0) \times \prod_{k=1}^{\ell} \left( \mathfrak{F}(A_{k-1}) \cap \mathfrak{T}(A_k) \right) \times \mathfrak{F}(A_{\ell}) \rightarrow \mathfrak{B}.$$

We define the character of a subset  $\mathfrak{X} \subseteq \mathfrak{B}$  as the formal series

$$P_{\mathfrak{X}}(t) = \sum_{\nu \in \mathbb{N}I} \text{Card } \mathfrak{X}(\nu) t^{\nu},$$

where  $\mathfrak{X}(\nu) = \mathfrak{X} \cap \mathfrak{B}(\nu)$  is the set of elements of weight  $\nu$  in  $\mathfrak{X}$ . We denote the multiplicity of a root  $\alpha$  by  $m_{\alpha}$ ; thus  $m_{\alpha} = 1$  if  $\alpha$  is real and  $m_{\alpha} = r$  if  $\alpha$  is imaginary.

**Proposition 7.7** *Let  $A \subseteq B$  be two biconvex subsets. Then*

$$P_{\mathfrak{F}(A) \cap \mathfrak{T}(B)} = \prod_{\alpha \in B \setminus A} \frac{1}{(1 - t^{\alpha})^{m_{\alpha}}}.$$

*Proof.* We first note that the proposition includes the formulas

$$P_{\mathfrak{T}(B)} = \prod_{\alpha \in B} \frac{1}{(1 - t^\alpha)^{m_\alpha}} \quad \text{and} \quad P_{\mathfrak{F}(A)} = \prod_{\alpha \in \Phi_+ \setminus A} \frac{1}{(1 - t^\alpha)^{m_\alpha}} \quad (7.2)$$

as the particular cases  $A = \emptyset$  or  $B = \Phi_+$ .

Consider a nondecreasing list of biconvex subsets  $\mathbf{A} = (A_0, \dots, A_\ell)$ . By construction,  $\Xi(\mathbf{A})$  restricts to a bijection

$$\prod_{k=1}^{\ell} \left( \mathfrak{F}(A_{k-1}) \cap \mathfrak{T}(A_k) \right) \cong \mathfrak{F}(A_0) \cap \mathfrak{T}(A_\ell)$$

whence an equality

$$P_{\mathfrak{F}(A_0) \cap \mathfrak{T}(A_\ell)} = \prod_{k=1}^{\ell} P_{\mathfrak{F}(A_{k-1}) \cap \mathfrak{T}(A_k)}.$$

Now let  $A \subseteq B$  be two finite biconvex subsets. We can find  $u \in W$  and a reduced decomposition  $v = s_{i_1} \cdots s_{i_\ell}$  such that  $\ell(uv) = \ell(u) + \ell$ ,  $A = N_u$  and  $B = N_{uv}$ . We apply the previous result to  $A_k = N_{us_{i_1} \cdots s_{i_k}}$ . Proposition 7.6 implies that  $P_{\mathfrak{F}(A_{k-1}) \cap \mathfrak{T}(A_k)}(t) = 1/(1 - t^{\beta_k})$ , where  $\beta_k = us_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$ , hence

$$P_{\mathfrak{F}(A) \cap \mathfrak{T}(B)}(t) = P_{\mathfrak{F}(A_0) \cap \mathfrak{T}(A_\ell)}(t) = \prod_{k=1}^{\ell} \frac{1}{1 - t^{\beta_k}} = \prod_{\alpha \in B \setminus A} \frac{1}{1 - t^\alpha},$$

as announced.

A similar reasoning shows the result in the case where both  $A$  and  $B$  are cofinite. Taking  $A = \emptyset$  or  $B = \Phi_+$  then gives (7.2) for  $B$  finite and  $A$  cofinite. By Remark 7.4, these formulas also hold if  $B$  is an increasing union of finite biconvex subsets and  $A$  is a decreasing intersection of cofinite biconvex subsets.

Since  $\mathfrak{B}$  indexes a basis for  $U(\mathfrak{n}_+)$ , we have

$$P_{\mathfrak{B}}(t) = \prod_{\alpha \in \Phi_+} \frac{1}{(1 - t^\alpha)^{m_\alpha}}, \quad (7.3)$$

by Kostant's partition function formula. On the other hand, given a biconvex subset  $A$ , the bijection  $\Xi(A)$  gives the equation  $P_{\mathfrak{B}} = P_{\mathfrak{T}(A)} \times P_{\mathfrak{F}(A)}$ . One of the two factors in the right-hand side is known to be given by (7.2) (since either  $\delta \in A$  or  $\delta \notin A$ ), we deduce that the other is also given by (7.2). So (7.2) is valid for any biconvex subsets  $A$  and  $B$ .

Now take  $A \subset B$ . The bijection

$$\mathfrak{T}(A) \times \left( \mathfrak{F}(A) \cap \mathfrak{T}(B) \right) \times \mathfrak{F}(B) \cong \mathfrak{B}$$

gives

$$P_{\mathfrak{F}(A) \cap \mathfrak{T}(B)} = \frac{P_{\mathfrak{B}}}{P_{\mathfrak{T}(A)} P_{\mathfrak{F}(B)}}.$$

The numerator in this fraction has been given just above, and the denominator is given by (7.2). Thus, we get the desired formula for  $P_{\mathfrak{F}(A) \cap \mathfrak{T}(B)}$ .  $\square$

In view of its later use, the particular case  $(A, B) = (A_{\theta}^{\min}, A_{\theta}^{\max})$  with  $\theta \in \mathfrak{t}$  deserves a special mention.

**Corollary 7.8** *Let  $\theta \in \mathfrak{t}$ . Then*

$$P_{\mathfrak{B}_{\theta}} = \left( \prod_{\substack{\alpha \in \Phi_+^{\text{re}} \\ \langle \theta, \alpha \rangle = 0}} \frac{1}{1 - t^{\alpha}} \right) \left( \prod_{n \geq 1} \frac{1}{1 - t^{n\delta}} \right)^r.$$

### 7.3 Simple regular modules

We come back to the description of the abelian categories  $\mathcal{R}_F$ . Our aim in this section is to get information on their simple objects.

We begin with a general remark: let  $F$  and  $G$  be two faces such that  $F \subseteq \overline{G}$ . If we pick  $\theta \in F$  and  $\eta \in G$ , then

$$A_{\theta}^{\min} \subseteq A_{\eta}^{\min} \subseteq A_{\eta}^{\max} \subseteq A_{\theta}^{\max},$$

hence

$$(\mathcal{I}_{\theta}, \overline{\mathcal{P}}_{\theta}) \preccurlyeq (\mathcal{I}_{\eta}, \overline{\mathcal{P}}_{\eta}) \preccurlyeq (\overline{\mathcal{I}}_{\eta}, \mathcal{P}_{\eta}) \preccurlyeq (\overline{\mathcal{I}}_{\theta}, \mathcal{P}_{\theta});$$

in other words,

$$(\mathcal{I}_F, \overline{\mathcal{P}}_F) \preccurlyeq (\mathcal{I}_G, \overline{\mathcal{P}}_G) \preccurlyeq (\overline{\mathcal{I}}_G, \mathcal{P}_G) \preccurlyeq (\overline{\mathcal{I}}_F, \mathcal{P}_F).$$

Therefore

$$\mathcal{I}_F \subseteq \mathcal{I}_G, \quad \mathcal{P}_F \subseteq \mathcal{P}_G \quad \text{and} \quad \mathcal{R}_F \supseteq \mathcal{R}_G,$$

and for any  $\Lambda$ -module  $T$ , we have a filtration  $0 \subseteq T_{\theta}^{\min} \subseteq T_{\eta}^{\min} \subseteq T_{\eta}^{\max} \subseteq T_{\theta}^{\max} \subseteq T$ . The three subquotients

$$T_{\eta}^{\min}/T_{\theta}^{\min} \in \overline{\mathcal{P}}_{\theta} \cap \mathcal{I}_{\eta}, \quad T_{\eta}^{\max}/T_{\eta}^{\min} \in \overline{\mathcal{P}}_{\eta} \cap \overline{\mathcal{I}}_{\eta} \quad \text{and} \quad T_{\theta}^{\max}/T_{\eta}^{\max} \in \mathcal{P}_{\eta} \cap \overline{\mathcal{I}}_{\theta}$$

all belong to  $\mathcal{R}_\theta$ ; in particular, a simple object of  $\mathcal{R}_F$  belongs either to  $\mathcal{I}_G$ ,  $\mathcal{R}_G$  or  $\mathcal{P}_G$ .

We denote by  $\text{Irr } \mathcal{R}_F$  the set of simple objects in  $\mathcal{R}_F$ . Recall that two objects  $T$  and  $U$  in  $\text{Irr } \mathcal{R}_F$  are said to be linked if there is a sequence  $T = X_0, X_1, \dots, X_n = U$  of objects in  $\text{Irr } \mathcal{R}_F$  such that  $\text{Ext}_\Lambda^1(X_{k-1}, X_k) \neq 0$  for each  $k \in \{1, \dots, n\}$ . (Note here that the groups  $\text{Ext}^1$  are the same computed in  $\Lambda\text{-mod}$  and in  $\mathcal{R}_F$ , for the latter is closed under extensions, and that  $\text{Ext}_\Lambda^1(X, Y)$  and  $\text{Ext}_\Lambda^1(Y, X)$  are  $K$ -dual to each other.) The linkage relation is an equivalence relation. Recall also that a module  $X$  is said to be rigid if  $\text{Ext}_\Lambda^1(X, X) = 0$ .

**Theorem 7.9** *Let  $F$  be a face of the spherical Weyl fan.*

- (i) *If the dimension-vector of a simple object  $T \in \mathcal{R}_F$  is a multiple of  $\delta$ , then  $T$  is alone in its linkage class in  $\text{Irr } \mathcal{R}_F$ , and  $T$  belongs to  $\mathcal{R}_C$  for each spherical Weyl chamber  $C$  such that  $F \subseteq \overline{C}$ .*
- (ii) *The other objects in  $\text{Irr } \mathcal{R}_F$  are rigid. Their dimension-vectors belong to  $\iota(\Phi^s)$ . Given  $\alpha \in \iota(\Phi^s)$ , there is at most one simple object in  $\mathcal{R}_F$  of dimension-vector  $\alpha$ , up to isomorphism.*

*Proof.* Let  $T$  be as in (i). For any  $X \in \text{Irr } \mathcal{R}_F$  different from  $T$ , we have  $\text{Hom}_\Lambda(T, X) = \text{Hom}_\Lambda(X, T) = 0$  by Schur's lemma, so  $\text{Ext}_\Lambda^1(T, X) = 0$  by Crawley-Boevey's formula (4.2). This shows that  $T$  is alone in its linkage class. Let  $C$  be a spherical Weyl chamber that contains  $F$  in its closure. The assumption on  $\underline{\dim} T$  rules out the possibility that  $T \in \mathcal{I}_C$  or  $\mathcal{P}_C$ . We conclude that  $T \in \mathcal{R}_C$ . Thus assertion (i) is true.

We now turn to (ii). Let  $T \in \text{Irr } \mathcal{R}_F$ , whose dimension-vector  $\alpha$  is not a multiple of  $\delta$ . Then  $(\alpha, \alpha)$  is a positive even integer. By Schur's lemma, the endomorphism algebra of  $T$  has dimension 1. Using Crawley-Boevey's formula (4.2), we then see that  $(\alpha, \alpha) = 2$  and  $\dim \text{Ext}_\Lambda^1(T, T) = 0$ . Thus  $T$  is a rigid  $\Lambda$ -module, and, by Proposition 5.10 in [30],  $\alpha$  is a real root.

Now, assume that  $\alpha - \delta$  is a positive root. By Proposition 7.7,  $\Lambda(\alpha - \delta)$  has an irreducible component whose general point belongs to  $\mathcal{R}_F$ . In particular, there exists  $X \in \mathcal{R}_F$  of dimension-vector  $\alpha - \delta$ . But then  $(\underline{\dim} X, \underline{\dim} T) = 2$ , and (4.2) gives that  $\text{Hom}_\Lambda(T, X)$  or  $\text{Hom}_\Lambda(X, T)$  is nonzero. Since  $\dim T > \dim X$ , this forbids  $T$  to be simple in  $\mathcal{R}_F$ , which contradicts our choice of  $T$ . Therefore  $\alpha - \delta \notin \Phi_+$ , which means that  $\alpha \in \iota(\Phi^s)$ .

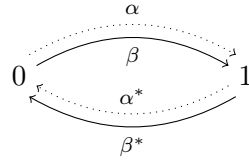
Lastly, let  $T'$  and  $T''$  be two simple objects in  $\mathcal{R}_F$  with the same dimension-vector  $\alpha \in \iota(\Phi^s)$ . Since  $(\underline{\dim} T', \underline{\dim} T'') = 2$ , (4.2) gives that  $\text{Hom}_\Lambda(T', T'')$  or  $\text{Hom}_\Lambda(T'', T')$  is nonzero. By Schur's lemma,  $T'$  and  $T''$  are isomorphic.  $\square$



One can show that the simple objects of  $\mathcal{R}_F$  described in Theorem 7.9 (i) always have dimension-vector  $\delta$  (see Corollary 7.23), and that Theorem 7.9 (ii) still holds when  $K$  is not algebraically closed.

## 7.4 The type $\tilde{A}_1$

Let  $\Pi$  be the completed preprojective algebra of the Kronecker quiver, as in Section 6. We identify the Grothendieck group of  $\Pi$ -mod with  $\mathbb{Z}^2$  by writing the dimension-vector of a  $\Pi$ -module  $V$  as the pair  $(\dim V_0, \dim V_1)$ .



As in Section 6.4, we denote by  $\Pi(\mu)$  the nilpotent variety of type  $\tilde{A}_1$  for a given dimension-vector  $\mu = (\mu_0, \mu_1)$ . A point in  $\Pi(\mu)$  is a 4-tuple of matrices  $T = (T_\alpha, T_\beta, T_{\alpha^*}, T_{\beta^*})$  that satisfy the equations  $T_\alpha T_{\alpha^*} + T_\beta T_{\beta^*} = 0$  and  $T_{\alpha^*} T_\alpha + T_{\beta^*} T_\beta = 0$  and that satisfy the nilpotency condition.

We denote the root system of type  $\tilde{A}_1$  by  $\Delta$ , so

$$\begin{aligned} \Delta_+ &= \{(n+1, n), (n, n+1), (n+1, n+1) \mid n \in \mathbb{N}\} \\ &= \{(1, 0) + n\delta, (0, 1) + n\delta, (n+1)\delta \mid n \in \mathbb{N}\} \end{aligned}$$

and  $\Delta_+ = \Delta_+^{\text{re}} \sqcup \mathbb{Z}_{>0} \delta$ , where  $\delta = (1, 1)$  is the indecomposable imaginary root. (Note that we use the same letter  $\delta$  to denote the indecomposable imaginary root in both  $\Phi_+$  and  $\Delta_+$ ; this will not lead to confusion.)

There are two opposite spherical chamber coweights, namely

$$\gamma'(\mu_0, \mu_1) = \mu_0 - \mu_1 \quad \text{and} \quad \gamma''(\mu_0, \mu_1) = \mu_1 - \mu_0.$$

The spherical Weyl fan has three faces, namely  $\{0\}$  and the two rays  $\mathbb{R}_{>0}\gamma'$  and  $\mathbb{R}_{>0}\gamma''$ .

Given  $n \in \mathbb{N}$ , we denote by  $\Pi(n\delta)^\times$  the open subset of all points  $T \in \Pi(n\delta)$  such that the  $n \times n$  matrix  $T_\alpha$  is invertible. Obviously, the dimension-vector  $\mu$  of a submodule of  $T$  satisfies  $\mu_0 \leq \mu_1$ , so  $T$  is  $\gamma'$ -semistable.

We now describe  $\Pi(n\delta)^\times$  with the help of an auxiliary variety. Let  $Z_n = \{(X, Y) \in M_n(K) \mid X \text{ is nilpotent, } XY = YX\}$ . Given a partition  $\lambda$  of size  $n$ , let us denote by  $\mathcal{O}_\lambda \subseteq M_n(K)$  the

adjoint orbit of nilpotent matrices of Jordan type  $\lambda$ . The following lemma is due to I. Frenkel and Savage (it is a particular case of [19], Proposition 2.9).

**Lemma 7.10** (i) *The map  $f : \Pi(n\delta)^\times \rightarrow Z_n$  defined by  $f(T) = (T_{\beta^*}T_\alpha, T_\alpha^{-1}T_\beta)$  is a principal  $\mathrm{GL}_n(K)$ -bundle.*

(ii) *The first projection  $Z_n \rightarrow M_n(K)$  identifies  $Z_n$  with the disjoint union of the conormal bundles  $T_{\mathcal{O}_\lambda}^*$ .*

*Proof.* We begin with (i). Let  $T \in \Pi(n\delta)^\times$ . By definition,  $T_{\beta^*}T_\alpha$  is nilpotent. In addition,

$$(T_{\beta^*}T_\alpha)(T_\alpha^{-1}T_\beta) = -T_{\alpha^*}T_\alpha = -T_\alpha^{-1}(T_\alpha T_{\alpha^*})T_\alpha = (T_\alpha^{-1}T_\beta)(T_{\beta^*}T_\alpha),$$

thanks to the preprojective equations. So  $f$  is well defined. Now  $\mathrm{GL}_n(K)$  acts on  $\Pi(n\delta)^\times$  by

$$U \cdot (T_\alpha, T_\beta, T_{\alpha^*}, T_{\beta^*}) = (UT_\alpha, UT_\beta, T_{\alpha^*}U^{-1}, T_{\beta^*}U^{-1}).$$

This action is free, for  $T_\alpha$  is invertible, and the orbits of this action are the fibers of  $f$ . Thus (i) holds true.

Assertion (ii) comes from the fact that under the standard trace duality, the commutant of a matrix  $X \in M_n(K)$  identifies with the orthogonal of the tangent space of the adjoint orbit going through  $X$ :

$$\begin{aligned} XY = YX &\iff \left( \forall W \in M_n(K), \operatorname{Tr}(W(XY - YX)) = 0 \right) \\ &\iff \left( \forall W \in M_n(K), \operatorname{Tr}([W, X]Y) = 0 \right). \end{aligned}$$

□

Thanks to Lemma 7.10, we see that the irreducible components of  $\Pi(n\delta)^\times$  are of the form  $f^{-1}(\overline{T_{\mathcal{O}_\lambda}^*})$ . We denote by  $I(\lambda)$  the closure in  $\Pi(n\delta)$  of this set; since  $\Pi(n\delta)^\times$  is open, this is an irreducible component of  $\Pi(n\delta)$ , whose general point is  $\gamma'$ -semistable. When  $\lambda$  has just one nonzero part, here  $n$ , we write simply  $I(n)$  instead of  $I((n))$ .

We denote the set of partitions by  $\mathcal{P}$ . Recalling the well-known formula

$$\sum_{\lambda \in \mathcal{P}} t^{|\lambda|} = \prod_{n \geq 1} \frac{1}{1 - t^n}$$

and applying Corollary 7.8 to  $\Pi$  and  $\gamma'$ , we then see that  $\{I(\lambda) \mid \lambda \in \mathcal{P}\}$  is the full set of elements in  $\mathfrak{R}_{\gamma'}$ .

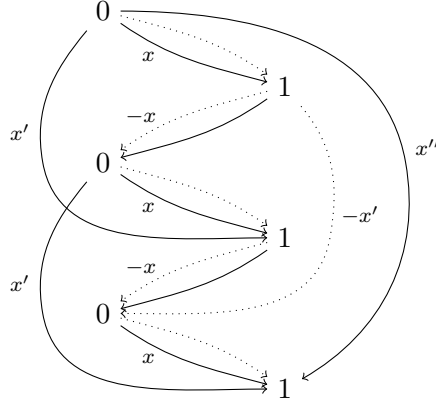


Figure 4: The general point of  $I(3)$ . All depicted arrows act by 1, except when otherwise indicated. One sees clearly that  $T_\alpha$  (represented by the dotted lines from vertices 0 to vertices 1) is an isomorphism and that  $T_{\beta*}T_\alpha$  is a Jordan block of size 3. The isomorphism class of  $T$  depends on 3 parameters  $x, x', x''$ .

**Proposition 7.11** *Let  $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell)$  be a partition.*

(i) *The canonical decomposition (in the sense of Section 4.4) of  $I(\lambda)$  is*

$$I(\lambda) = \overline{I(\lambda_1) \oplus \dots \oplus I(\lambda_\ell)}.$$

(ii) *The component  $I(\lambda)^*$  is obtained from  $I(\lambda)$  by applying the automorphism of  $\Pi$  that exchanges the vertices 0 and 1, the arrows  $\alpha$  and  $\alpha^*$ , and the arrows  $\beta$  and  $\beta^*$ .*

*Proof.* If  $S'$  and  $S''$  are two points in  $I(1)$ , then they are distinct simple objects in the category  $\mathcal{B}_{\gamma'}$ , so  $\text{Hom}_\Pi(S', S'') = \text{Hom}_\Pi(S'', S') = 0$  by Schur's lemma. Using Crawley-Boevey's formula (4.2), we deduce that  $\text{Ext}_\Pi^1(S', S'') = 0$ . Since the general point of  $I(n)$  is the  $n$ -th iterated extension of a general point in  $I(1)$ , we deduce that  $\text{ext}_\Pi^1(I(m), I(n)) = 0$  for all  $(m, n) \in \mathbb{N}^2$ . Moreover, the general point of  $I(n)$  is an indecomposable  $\Pi$ -module. From all this, it follows that for any partition  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ , the closure  $Z = \overline{I(\lambda_1) \oplus \dots \oplus I(\lambda_\ell)}$  is an irreducible component of  $\Pi(n|\lambda|)$ . Up to isomorphism, a general point  $T \in Z$  can then be depicted as the direct sum of modules of the kind represented on Figure 4. From such a representation, one deduces that the nilpotent map  $T_{\beta*}T_\alpha$  has Jordan type  $\lambda$ . Therefore  $Z = I(\lambda)$ , as asserted in (i).

We thus know how to find a nice representative in a general orbit in  $I(\lambda)$ . To compute  $I(\lambda)^*$ , we can thus apply the  $*$ -duality on this representative; applying thereafter the automorphism

of  $\Pi$  described in statement (ii), we end up with  $T_\alpha, T_\beta, T_{\alpha^*}, T_{\beta^*}$  replaced by their transposed maps. A comparison with the picture on Figure 4 shows that we land in  $I(\lambda)$ . Thus, on the component  $I(\lambda)$ , applying the  $*$  duality gives the same result as applying our automorphism, as claimed in (ii).  $\square$

Assertion (ii) in this proposition implies that  $\{I(\lambda)^* \mid \lambda \in \mathcal{P}\}$  is the full set of elements in  $\mathfrak{R}_\gamma''$ .

## 7.5 Cores

Let  $\gamma \in \Gamma$  be a chamber coweight. We define the category of  $\gamma$ -cores as the intersection

$$\bigcap_{\substack{F \text{ face} \\ \gamma \in \overline{F}}} \mathcal{R}_F = \bigcap_{\substack{C \text{ Weyl chamber} \\ \gamma \in \overline{C}}} \mathcal{R}_C.$$

This is an abelian, closed under extensions, subcategory of  $\Lambda\text{-mod}$ . The dimension-vector of a  $\gamma$ -core is a multiple of  $\delta$ .

The following proposition provides an alternative definition of  $\gamma$ -cores.

**Proposition 7.12** *A object in  $\mathcal{R}_\gamma$  is a  $\gamma$ -core if and only if the dimension-vectors of all its Jordan-Hölder components are multiple of  $\delta$ .*

*Proof.* By Theorem 7.9 (i), a simple  $\mathcal{R}_\gamma$ -module whose dimension-vector is multiple of  $\gamma$  is necessarily a  $\gamma$ -core. The sufficiency of the condition follows then from the fact that the category of  $\gamma$ -cores is closed under extensions.

Conversely, let  $T \in \mathcal{R}_\gamma$ . In view of the description of the linkage classes of  $\mathcal{R}_\gamma$ , if  $T$  has a Jordan-Hölder component whose dimension-vector is not a multiple of  $\delta$ , then such a component can be found in the socle of  $T$  (as an object in  $\mathcal{R}_\gamma$ ). Thus  $T$  contains a submodule  $X$  such that  $\underline{\dim} X$  is in the kernel of  $\gamma$  but is not multiple of  $\delta$ , so there exists  $\theta$  near  $\gamma$ , in a Weyl chamber, such that  $\langle \theta, \underline{\dim} X \rangle > 0$ . Then  $T$  is not in  $\mathcal{R}_\theta$ , so it is not a  $\gamma$ -core. This proves the necessity of the condition given in the lemma.  $\square$

Obviously, the  $*$ -dual of a  $\gamma$ -core is a  $(-\gamma)$ -core. More interesting is the following compatibility between cores and reflection functors.

**Proposition 7.13** *Let  $\gamma \in \Gamma$ , let  $i \in I$  and let  $T$  be a  $\gamma$ -core. If  $\langle \gamma, \alpha_i \rangle > 0$ , then  $T \in \mathcal{T}^{s_i}$  and  $\Sigma_i T$  is a  $s_i \gamma$ -core. If  $\langle \gamma, \alpha_i \rangle < 0$ , then  $T \in \mathcal{T}_{s_i}$  and  $\Sigma_i^* T$  is a  $s_i \gamma$ -core. If  $\langle \gamma, \alpha_i \rangle = 0$ , then  $T \cong \Sigma_i T \cong \Sigma_i^* T$ .*

*Proof.* The first two claims immediately follow from Theorem 5.18, so let us consider the case where  $\langle \gamma, \alpha_i \rangle = 0$ . Then there exists  $\theta \in \mathfrak{t}$  close to  $\gamma$  such that  $\langle \theta, \alpha_i \rangle > 0$ , which forbids  $S_i$  to appear as a submodule of  $T$ , and there exists  $\eta \in \mathfrak{t}$  close to  $\gamma$  such that  $\langle \eta, \alpha_i \rangle < 0$ , which forbids  $S_i$  to appear as a quotient of  $T$ . Therefore the  $i$ -socle and the  $i$ -head of  $T$  are both trivial. Now recall the diagram (4.1). We rewrite it as

$$T_i \xrightarrow{T_{\text{out}(i)}} \tilde{T}_i \xrightarrow{T_{\text{in}(i)}} T_i.$$

This is a complex,  $T_{\text{out}(i)}$  is injective, and  $T_{\text{in}(i)}$  is surjective. We have  $(\alpha_i, \underline{\dim} T) = 0$ , for  $\underline{\dim} T$  is a multiple of  $\delta$ , so the dimension of  $\tilde{T}_i$  is twice the dimension of  $T_i$ . Our complex is therefore a short exact sequence. The isomorphisms  $T \cong \Sigma_i T \cong \Sigma_i^* T$  then follow from Proposition 5.1.  $\square$

We will produce  $\gamma$ -cores by means of Hall functors. According to Section 6, we need to construct pairs  $(S, R)$  of  $\Lambda$ -modules that satisfy (6.1). We proceed as follows.

We call flag a pair  $(C, F)$  consisting of a Weyl chamber  $C$  and a facet  $F$  contained in the closure of  $C$ . Such a pair determines a chamber coweight  $\gamma_{C/F}$ , defined by the equation  $C = F + \mathbb{R}_{>0} \gamma_{C/F}$ . The spherical Weyl group  $W_0$  acts on the set of flags.

Take a flag  $(C, F)$  and pick  $\theta \in F$ . Then  $\Phi \cap (\ker \theta)$  is an affine root system of type  $\tilde{A}_1$  and  $\Phi^s \cap (\ker \theta)$  is a root system of type  $A_1$ , consisting of two opposite roots  $\pm\alpha$ . By Theorem 7.9, the dimension-vectors of the simple objects in  $\mathcal{R}_F$  belong to  $\{\iota(\alpha), \iota(-\alpha)\} \cup \mathbb{Z}_{>0} \delta$ . This implies that an object in  $\mathcal{R}_F$  of dimension-vector  $\iota(\pm\alpha)$  is necessarily simple.

Now Corollary 7.8 guarantees the existence of a whole irreducible component of  $\Lambda(\iota(\pm\alpha))$  whose general point is in  $\mathcal{R}_F$ . Therefore there are objects in  $\mathcal{R}_F$  of dimension-vector  $\iota(\pm\alpha)$ . These objects are necessarily simple, and therefore unique up to isomorphism, by Theorem 7.9 (ii). We denote them by  $S_{C,F}$  and  $R_{C,F}$ , the labels being adjusted so that

$$\langle \gamma_{C/F}, \underline{\dim} S_{C,F} \rangle = 1 \quad \text{and} \quad \langle \gamma_{C/F}, \underline{\dim} R_{C,F} \rangle = -1. \quad (7.4)$$

*Examples 7.14.* (i) One fashion to produce  $S_{C,F}$  and  $R_{C,F}$  is to use Proposition 7.6: if we set  $A = A_\theta^{\min} = \{\alpha \in \Phi_+ \mid \langle \theta, \alpha \rangle > 0\}$ ,  $B' = A \sqcup \{\iota(\alpha)\}$  and  $B'' = A \sqcup \{\iota(-\alpha)\}$ , then both modules  $L(A, B')$  and  $L(A, B'')$  belong to  $\mathcal{R}_\theta$  and have the correct dimension-vectors to be  $S_{C,F}$  and  $R_{C,F}$ . Alternatively, we can consider  $L(A', B)$  and  $L(A'', B)$ , where  $B = A_\theta^{\max} = \{\alpha \in \Phi_+ \mid \langle \theta, \alpha \rangle \geq 0\}$ ,  $A' = B \setminus \{\iota(\alpha)\}$  and  $A'' = B \setminus \{\iota(-\alpha)\}$ .

(ii) A facet  $F$  separates two chambers, say  $C'$  and  $C''$ . We then have  $(S_{C',F}, R_{C',F}) = (R_{C'',F}, S_{C'',F})$ .

- (iii) If  $(C, F)$  is a flag, then  $(-C, -F)$  is also a flag, and we have  $(S_{-C, -F}, R_{-C, -F}) = ((R_{C, F})^*, (S_{C, F})^*)$ .
- (iv) Let us fix a zero node  $0 \in I$ , as explained at the end of Section 2.2. Then each  $i \in I_0$  provides a flag  $(C_0^s, F_{\{i\}})$ , where

$$F_{\{i\}} = \{\theta \in \mathfrak{t} \mid \langle \theta, \alpha_i \rangle = 0, \langle \theta, \alpha_j \rangle > 0 \text{ for all } j \in I_0 \setminus \{i\}\}.$$

We have  $\gamma_{C_0^s/F_{\{i\}}} = \varpi_i$  and  $\{\iota(\pm\alpha)\} = \{\alpha_i, \delta - \alpha_i\}$ . The dimension-vector of  $S_{C_0^s, F_{\{i\}}}$  is  $\alpha_i$ , hence  $S_{C_0^s, F_{\{i\}}} = S_i$ . We define  $R_i = R_{C_0^s, F_{\{i\}}}$ . Since  $R_i$  is in  $\mathcal{R}_{F_{\{i\}}}$ , it cannot contain any submodule isomorphic to  $S_j$ , with  $j \in I_0 \setminus \{i\}$ , and since it is simple, it cannot contain any submodule isomorphic to  $S_i$  either. Therefore  $R_i$  has dimension-vector  $\delta - \alpha_i$  and its socle is  $S_0$ . By Lemma 2 (2) of [16], these two conditions characterize  $R_i$ .

- (v) Keeping our zero node  $0 \in I$ , let  $(i, w) \in I_0 \times W_0$  be such that  $\ell(ws_i) > \ell(w)$  and consider  $(C, F) = (wC_0^s, wF_{\{i\}})$ . By Theorem 5.19 (i), we have an equivalence of categories

$$\mathcal{R}_{F_{\{i\}}} \xrightleftharpoons[\text{Hom}_\Lambda(I_w, ?)]{I_w \otimes \Lambda} \mathcal{R}_{wF_{\{i\}}},$$

which carries  $(S_i, R_i)$  to  $(S_{C, F}, R_{C, F})$ .

**Lemma 7.15** *The modules  $S_{C, F}$  and  $R_{C, F}$  satisfy the conditions (6.1).*

*Proof.* The modules  $S_{C, F}$  and  $R_{C, F}$  are simple objects in  $\mathcal{R}_F$ , so by Schur's lemma, they are orthogonal bricks:

$$\text{End}_\Lambda(S_{C, F}) = \text{End}_\Lambda(R_{C, F}) = K, \quad \text{Hom}_\Lambda(S_{C, F}, R_{C, F}) = \text{Hom}_\Lambda(R_{C, F}, S_{C, F}) = 0.$$

The remaining equations

$$\text{Ext}_\Lambda^1(S_{C, F}, S_{C, F}) = \text{Ext}_\Lambda^1(R_{C, F}, R_{C, F}) = 0, \quad \dim \text{Ext}_\Lambda^1(S_{C, F}, R_{C, F}) = 2$$

follow from Crawley-Boevey's formula (4.2).  $\square$

We can thus apply the results of Section 6 to the modules  $S_{C, F}$  and  $R_{C, F}$ . We get a Hall functor  $\mathcal{H}_{C, F} : \Pi\text{-mod} \rightarrow \Lambda\text{-mod}$ , which is an equivalence of categories between  $\Pi\text{-mod}$  and the subcategory  $\langle S_{C, F}, R_{C, F} \rangle$  of  $\mathcal{R}_F$ .

We denote by  $\mathfrak{A} = \bigsqcup_{\mathbf{d} \in \mathbb{N}^2} \text{Irr } \Pi(\mathbf{d})$  the analog of the crystal  $\mathfrak{B}$ , but for the Kronecker quiver. Proposition 6.4 claims that  $\mathcal{H}_{C, F}$  induces an injection  $\mathfrak{H}_{C, F} : \mathfrak{A} \rightarrow \mathfrak{B}$ , whose image consists of those components whose general points lie in  $\langle S_{C, F}, R_{C, F} \rangle$ .

For  $\lambda$  a partition, recall the irreducible component  $I(\lambda) \in \mathfrak{A}$  defined in Section 7.4. We will see that the general point of  $\mathfrak{H}_{C,F}(I(\lambda))$  is a  $\gamma_{C,F}$ -core, and moreover that  $\mathfrak{H}_{C,F}(I(\lambda))$  depends only on  $\gamma_{C,F}$  and on  $\lambda$ .

To establish these results, our strategy is to first focus on the particular case described in Example 7.14 (iv). So now let us fix a zero node  $0 \in I$ , let us take  $i \in I_0$ , and let us set  $I(\varpi_i, \lambda) = \mathfrak{H}_{C_0^s, F_{\{i\}}}(I(\lambda))$ . The module  $R_i$  is rigid, hence the closure of its orbit in  $\Lambda(\delta - \alpha_i)$  is an irreducible component, which we denote by  $Z_i$ . We note that  $I(\varpi_1, 1) = \tilde{e}_i Z_i$ . The following proposition is essentially a reformulation of [16], Theorem 2.

**Proposition 7.16** (i) One has  $\mathfrak{R}_{C_0^s}(\delta) = \{I(\varpi_i, 1) \mid i \in I_0\}$ .

(ii) For each  $i \in I_0$ , a general point in  $I(\varpi_i, 1)$  has socle  $S_0$  and head  $S_i$ .

(iii) For each  $i \in I_0$ , a general point in  $I(\varpi_i, 1)$  is a  $\varpi_i$ -core. Conversely, any  $\varpi_i$ -core of dimension-vector  $\delta$  belongs to  $I(\varpi_i, 1)$ .

*Proof.* We first note that the socle of a module  $T \in \mathcal{R}_{C_0^s}$  is necessarily concentrated at the vertex 0, since a submodule of  $T$  of dimension-vector  $\alpha_i$  with  $i \in I_0$  would destabilize  $T$ . If  $\underline{\dim} T = \delta$ , then  $\text{soc } T \cong S_0$ .

Let  $\Lambda_0 = \{T \in \Lambda(\delta) \mid T \in \mathcal{R}_{C_0^s}\}$ , an open subset of  $\Lambda(\delta)$ . Then  $\mathfrak{R}_{C_0^s}(\delta)$  can be identified with the set of irreducible components of  $\Lambda_0$ .

Let  $T \in \Lambda_0$  and let  $S_i$  in the head of  $T$ . There is then a surjective morphism  $T \rightarrow S_i$ . Its kernel has socle  $S_0$  and dimension-vector  $\delta - \alpha_i$ , so is isomorphic to  $R_i$ . We thus have a short exact sequence  $0 \rightarrow R_i \rightarrow T \rightarrow S_i \rightarrow 0$ , which shows that  $T$  belongs to  $\tilde{e}_i Z_i = I(\omega_i, 1)$ .

Therefore  $\Lambda_0$  is covered by the irreducible components  $I(\omega_i, 1)$ . Now Corollary 7.8 asserts that  $\Lambda_0$  has  $r$  irreducible components. Assertion (i) follows.

If a point in  $\Lambda_0$  has two different simple modules  $S_i$  and  $S_j$  in its head, then it belongs to both  $I(\omega_i, 1)$  and  $I(\omega_j, 1)$ , and therefore is not general. This shows (ii).

Now fix  $i \in I_0$ . Let  $T$  be a general point in  $I(\varpi_i, 1)$ . Then  $\varpi_i(\underline{\dim} T) = 0$  and (ii) implies that  $\varpi_i(\underline{\dim} X) < 0$  for any proper submodule  $X \subseteq T$ . The module  $T$  is thus a simple object in  $\mathcal{R}_{\varpi_i}$ . By Theorem 7.9 (i),  $T$  is a  $\varpi_i$ -core.

Conversely, if  $T$  is a  $\varpi_i$ -core of dimension-vector  $\delta$ , then its socle must be  $S_0$  and only  $S_i$  can show up in the head of  $T$ . The reasoning used to prove (i) applies anew, and we conclude that  $T$  belongs to  $I(\varpi_i, 1)$ . (More exactly, the  $G(\delta)$ -orbit of points in  $\Lambda(\delta)$  isomorphic to  $T$  is contained in  $I(\varpi_i, 1)$ .)  $\square$

*Remark 7.17.* With the notation of the proof, a point  $T$  in  $\Lambda_0 \cap I(\varpi_i, 1)$  is always the middle term of a non-split extension  $0 \rightarrow R_i \rightarrow T \rightarrow S_i \rightarrow 0$ . Using the fact that  $\dim \operatorname{Ext}_\Lambda^1(S_i, R_i) = 2$ , one shows that the datum of the isomorphism class of  $T$  is equivalent to the datum of the class of the extension, up to scalar. In other words,  $G(\delta)$ -orbits in  $\Lambda_0 \cap I(\varpi_i, 1)$  are in bijection with points in the projective line  $\mathbb{P}(\operatorname{Ext}_\Lambda^1(S_i, R_i))$ . In [16], Crawley-Boevey shows that in the moduli space  $\Lambda_0 // G(\delta)$  (the quotient here should be understood in the GIT sense w.r.t. a character  $\theta \in C_0^s$ , see [36], Definition 2.1), these projective lines intersect as displayed by the edges of the Dynkin diagram.

**Lemma 7.18** (i) *Let  $i \in I_0$ . For any partition  $\lambda$ , a general point in  $I(\varpi_i, \lambda)$  is a  $\varpi_i$ -core.*  
(ii) *Let  $(i, n) \in I_0 \times \mathbb{N}$ . A general point in  $I(\varpi_i, n)$  is indecomposable in  $\Lambda$ -mod.*  
(iii) *For  $(i, m)$  and  $(j, n)$  in  $I_0 \times \mathbb{N}$ , we have  $\operatorname{ext}_\Lambda^1(I(\varpi_i, m), I(\varpi_j, n)) = 0$ . In addition, if  $(i, m) \neq (j, n)$ , then  $I(\varpi_i, m) \neq I(\varpi_j, n)$ .*

*Proof.* A general point  $T$  of  $I(\lambda)$  is an extension of elements  $J_1, \dots, J_n$  in  $I(1)$ . Moreover, the  $J_k$  can be chosen in an arbitrary open subset of  $I(1)$ . The general point  $\mathcal{H}_{C_0^s, F_{\{i\}}}(T)$  of  $I(\varpi_i, n)$  is thus an extension of the modules  $\mathcal{H}_{C_0^s, F_{\{i\}}}(J_k)$ . By Proposition 7.16 (iii), each  $\mathcal{H}_{C_0^s, F_{\{i\}}}(J_k)$  is a  $\varpi_i$ -core, so  $\mathcal{H}_{C_0^s, F_{\{i\}}}(T)$  is also a  $\varpi_i$ -core. This shows (i).

When  $\lambda$  has just one part, say  $\lambda = (n)$ , a general point  $T$  of  $I(n)$  is indecomposable in  $\Pi$ -mod, as we observed in the proof of Proposition 7.11. This indecomposability is preserved by the full and faithful functor  $\mathcal{H}_{C_0^s, F_{\{i\}}}$ , whence (ii).

By Proposition 7.16 (ii), the head of a general point in  $I(\varpi_i, 1)$  is concentrated at vertex  $i$ . The proof of (i) implies that the same is true for the general point in  $I(\varpi_i, n)$ . Therefore  $I(\varpi_i, n) \neq I(\varpi_j, n)$  if  $i \neq j$ . Moreover, two points  $T \in I(\varpi_i, 1)$  and  $X \in I(\varpi_j, 1)$  picked at random are simple non-isomorphic objects in  $\mathcal{R}_{C_0^s}$  (even if  $i = j$ ). By Schur's lemma, we have  $\operatorname{Hom}_\Lambda(T, X) = \operatorname{Hom}_\Lambda(X, T) = 0$ , and so  $\operatorname{Ext}_\Lambda^1(T, X) = 0$  by Crawley-Boevey's formula (4.2). Looking at the proof of (i), we see that this also holds true if  $(T, X)$  is picked at random in  $I(\varpi_i, m) \times I(\varpi_j, n)$ . This shows assertion (iii).  $\square$

Any chamber weight  $\gamma$  can be written as  $w\varpi_i$ , with  $(i, w) \in I_0 \times W_0$ . We denote by  $I(\gamma, \lambda)$  the image of  $I(\varpi_i, \lambda)$  by the functor  $I_w \otimes ?$ , viewed as operating on irreducible components as in Section 5.5 (see in particular Propositions 5.7 and 5.24). Proposition 7.13 shows that  $I(\gamma, \lambda)$  does not depend on the choice of  $w$ , which justifies the notation; it also shows that a general point of  $I(\gamma, \lambda)$  is a  $\gamma$ -core.

Recall that the set of partitions is denoted by  $\mathcal{P}$ .



**Theorem 7.19** *Let  $C$  be a spherical Weyl chamber. Then the map*

$$(\lambda_\gamma) \mapsto \overline{\bigoplus_{\gamma \in \Gamma \cap \overline{C}} I(\gamma, \lambda_\gamma)}$$

*is a bijection from  $\mathcal{P}^{\Gamma \cap \overline{C}}$  onto  $\mathfrak{R}_C$ .*

*Proof.* Using the reflection functors (specifically, Theorem 5.19 (i) and Propositions 5.7 and 5.24), we can reduce to the case of the dominant chamber  $C_0^s$ .

Lemma 7.18 shows that each  $\overline{\bigoplus_{\gamma \in \Gamma \cap \overline{C}_0^s} I(\gamma, \lambda_\gamma)}$  is an irreducible component whose general point belongs to  $\mathcal{R}_{C_0^s}$ . The canonical decomposition of these components show that they are pairwise distinct. The map described in the statement is thus well defined and injective. Its surjectivity follows from Corollary 7.8.  $\square$

Theorem 7.19 provides the following intrinsic characterization of  $I(\gamma, n)$ .

**Corollary 7.20** *For  $\gamma \in \Gamma$  and each  $n \geq 1$ ,  $I(\gamma, n)$  is the unique irreducible component of  $\Lambda(n\delta)$  whose general point is an indecomposable  $\gamma$ -core.*

This characterization proves that the components  $I(\gamma, n)$ , and thus also

$$I(\gamma, \lambda) = \overline{I(\gamma, \lambda_1) \oplus \cdots \oplus I(\gamma, \lambda_\ell)}$$

do not depend on the various choices made to construct them, notably in the construction of the functors  $\mathcal{H}_{C_0^s, F_{\{i\}}}$  used to define  $I(\varpi_i, \lambda)$ . It also implies that  $I(-\gamma, n) = I(\gamma, n)^*$ , hence

$$I(-\gamma, \lambda) = I(\gamma, \lambda)^* \tag{7.5}$$

for any  $(\gamma, \lambda) \in \Gamma \times \mathcal{P}$ .

This independence on the choice of the Hall functor leads to the following result.

**Corollary 7.21** *For all flags  $(C, F)$  and all partitions  $\lambda$ , we have  $\mathfrak{H}_{C, F}(I(\lambda)) = I(\gamma_{C, F}, \lambda)$ .*

*Proof.* Suppose that we are in the situation of Example 7.14 (v) with  $(C, F) = (wC_0^s, wF_{\{i\}})$ . Then  $\text{Hom}_\Lambda(I_w, ?) \circ \mathcal{H}_{C, F}$  is a Hall functor built from the datum of  $S_i$  and  $R_i$ , so it can play the role of  $\mathcal{H}_{C_0^s, F_{\{i\}}}$ . Thus by the above observation, the map on components defined by the functor  $\text{Hom}_\Lambda(I_w, ?)$  sends  $\mathfrak{H}_{C, F}(I(\lambda))$  to  $I(\varpi_i, \lambda)$ , for each partition  $\lambda$ . This immediately yields  $\mathfrak{H}_{C, F}(I(\lambda)) = I(\gamma_{C, F}, \lambda)$ .

This analysis applies to exactly half of the flags. The remaining flags are of the form  $(C, F) = (-wC_0^s, -wF_{\{i\}})$ ; their cases are deduced from the previous situation by duality.  $\square$

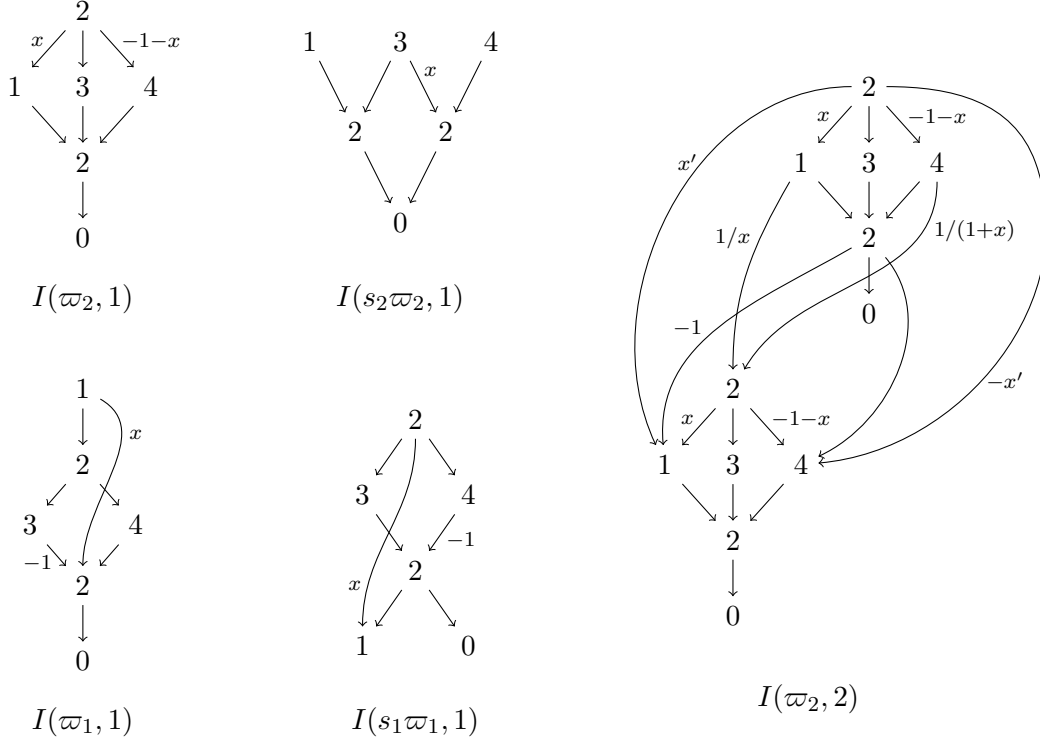


Figure 5: Examples of cores in type  $D_4$  (convention: the central node of the Dynkin diagram is 2). The isomorphism class of the general point of  $I(\gamma, n)$  depends on  $n$  parameters; here these parameters are called  $x$  and  $x'$ .

Combining the above result with Proposition 7.11 (ii) and Example 7.14 (ii), we obtain:

**Corollary 7.22** *Let  $F$  be a facet and let  $C'$  and  $C''$  be the two Weyl chambers that  $F$  separates. For any partition  $\lambda$ , we have  $\mathfrak{H}_{C',F}(I(\lambda)^*) = I(\gamma_{C''/F}, \lambda)$ .*

Our last corollary to Theorem 7.19 rounds off Theorem 7.9 (i).

**Corollary 7.23** *Let  $F$  be a face of the spherical Weyl fan. Let  $T$  be a simple object of  $\mathcal{R}_F$ . If  $\dim T$  is a multiple of  $\delta$ , then in fact  $\dim T = \delta$ .*

*Proof.* Let  $n$  be a positive integer. We denote by  $\Lambda(n\delta)^{(<n)}$  the open set of all points in  $\Lambda(n\delta)$  whose stabilizer with respect to the action of  $G(n\delta)$  has dimension smaller than  $n$ . Let  $C$  be a

Weyl chamber such that  $F \subseteq \overline{C}$ . Given  $Z \in \mathfrak{R}_C$  of weight  $n\delta$ , the description of Theorem 7.19 leads to the equality  $\dim \text{End}_\Lambda(T) = n$  for any general point  $T$  in  $Z$ ; therefore  $\Lambda(n\delta)^{(<n)}$  does not meet  $Z$ . Since  $Z$  was arbitrary in  $\mathfrak{R}_C$ , this means that no point in  $\Lambda(n\delta)^{(<n)}$  belongs to  $\mathcal{R}_C$ . The corollary now follows from Theorem 7.9 (i) and from Schur's lemma.  $\square$

**Theorem 7.24** *For any flag  $(C, F)$ , the map*

$$(Z, (\lambda_\gamma)) \mapsto \overline{\mathfrak{H}_{C,F}(Z)} \oplus \bigoplus_{\gamma \in \Gamma \cap \overline{F}} I(\gamma, \lambda_\gamma)$$

*is a bijection from  $\mathfrak{A} \times \mathcal{P}^{\Gamma \cap \overline{F}}$  onto  $\mathfrak{R}_F$ .*

*Proof.* By construction, the modules  $S_{C,F}$  and  $R_{C,F}$  are simple objects in  $\mathcal{R}_F$ . In addition, given  $\gamma \in \Gamma \cap \overline{F}$ , if  $X$  is a general point in  $I(\gamma, 1)$ , then  $X$  is a  $\gamma$ -core of dimension-vector  $\delta$  by Proposition 7.16, so  $X$  is simple in  $\mathcal{R}_\gamma$  by Proposition 7.12, so  $X$  is simple in  $\mathcal{R}_F$ . For dimension reasons,  $X$  is different from  $R_{C,F}$  and  $S_{C,F}$ , so the homomorphism spaces between  $X$  and  $R_{C,F}$  and between  $X$  and  $S_{C,F}$  are all zero, by Schur's lemma. Since  $\underline{\dim} X = \delta$ , we conclude from Crawley-Boevey's formula (4.2) that  $\text{Ext}_\Lambda^1(X, S_{C,F}) = \text{Ext}_\Lambda^1(X, R_{C,F}) = 0$ . The same kind of arguments as in the proof of Lemma 7.18 (iii) show then that  $\text{ext}_\Lambda^1(\mathfrak{H}_{C,F}(Z), I(\gamma, n)) = 0$  for any  $Z \in \mathfrak{A}$ . In view of Crawley-Boevey and Schröer's theory (see Section 4.4), this implies that our map is well defined.

Let  $Z \in \mathfrak{A}$ . If a component  $I(\gamma, n)$  occurred in the canonical decomposition of  $\mathfrak{H}_{C,F}(Z)$ , then a general point  $T$  of  $\mathfrak{H}_{C,F}(Z)$  would have a composition factor (in the abelian category  $\mathcal{R}_F$ ) isomorphic to an  $X$  as above, which is impossible because  $T$  belongs to the category  $\langle S_{C,F}, R_{C,F} \rangle$ . The set of indecomposable irreducible components that arise from the  $\mathfrak{H}_{C,F}(Z)$  is thus disjoint from  $\{I(\gamma, n) \mid \gamma \in \Gamma \cap \overline{F}, n \in \mathbb{N}\}$ . The uniqueness of the canonical decomposition of an element in  $\mathfrak{R}_F$  entails then that our map is injective.

Finally, we use a counting argument to prove the surjectivity of our map. Pick  $\theta \in F$ . If  $Z \in \mathfrak{A}$  has weight  $\mu = (\mu_0, \mu_1)$ , then  $\mathfrak{H}_{C,F}(Z) \in \mathfrak{B}$  has weight  $K_0(\mathcal{H}_{C,F})(\mu) = \mu_0 \underline{\dim} S_{C,F} + \mu_1 \underline{\dim} R_{C,F}$ . We here note that  $K_0(\mathcal{H}_{C,F})$  maps the imaginary root  $\delta \in \Delta_+$ , given by  $\mu_0 = \mu_1 = 1$ , to  $\underline{\dim} S_{C,F} + \underline{\dim} R_{C,F}$ , which is equal to  $\delta \in \Phi_+$ , the imaginary root in  $\Phi_+$ ; thus the use of the same notation for  $\delta$  for both root systems does not lead to any confusion. Plugging this information into the character series for  $\mathfrak{A}$ , given by the analog for  $\Delta_+$  of (7.3), and adding the contribution of the  $I(\gamma, \lambda_\gamma)$ , we can compute the character of the image of our map:

$$\prod_{n \in \mathbb{N}} \left( \frac{1}{(1 - t^{\underline{\dim} S_{C,F} + n\delta})} \times \frac{1}{(1 - t^{\underline{\dim} R_{C,F} + n\delta})} \times \frac{1}{(1 - t^{(n+1)\delta})} \right) \times \prod_{\gamma \in \Gamma \cap \overline{F}} \left( \sum_{\lambda_\gamma \in \mathcal{P}} t^{|\lambda_\gamma| \delta} \right).$$

This is equal to

$$P_{\mathfrak{R}_\theta} = \left( \prod_{\substack{\alpha \in \Phi_+^{\text{re}} \\ \langle \theta, \alpha \rangle = 0}} \frac{1}{1 - t^\alpha} \right) \left( \prod_{n \geq 1} \frac{1}{1 - t^{n\delta}} \right)^r,$$

which ensures that our map is surjective.  $\square$

Let  $(Z, (\lambda_\gamma)) \in \mathfrak{A} \times \mathcal{P}^{\Gamma \cap \overline{F}}$  and denote by  $\tilde{Z} \in \mathcal{R}_F$  its image under the map in Theorem 7.24. Pick general points  $X \in Z$  and  $X_\gamma \in I(\gamma, \lambda_\gamma)$ , for each  $\gamma \in \Gamma \cap \overline{F}$ . Then  $\tilde{X} = \mathcal{H}_{C,F}(X) \oplus \left( \bigoplus_\gamma X_\gamma \right)$  is a general point of  $\tilde{Z}$ . By Remark 3.5 (ii) (applied to the category  $\mathcal{R}_F$ ), the HN polytope of  $\tilde{X}$  is the Minkowski sum of the HN polytopes of its summands, that is, the Minkowski sum of  $K_0(\mathcal{H}_{C,F})(\text{Pol}(X))$  and of segments that join 0 to  $|\lambda_\gamma|\delta$ .

Pick now  $\theta \in F$  and  $\Lambda_b \in \mathfrak{B}$ . The bijection  $\Xi_\theta^{-1}$  maps  $\Lambda_b$  to say  $(\Lambda_{b'}, \Lambda_{b''}, \Lambda_{b'''}) \in \mathfrak{I}_F \times \mathfrak{R}_F \times \mathfrak{P}_F$ . If  $T \in \Lambda_b$  is a general point, then  $T_\theta^{\max}/T_\theta^{\min}$  is a general point in  $\Lambda_{b''}$ . Corollary 3.3 then says that the HN polytope of  $T_\theta^{\max}/T_\theta^{\min}$ , regarded as an object of  $\mathcal{R}_F$ , is the 2-face defined by  $\theta$  of the HN polytope of  $T$ . Putting  $\tilde{Z} = \Lambda_{b''}$  in the previous paragraph, we see that this 2-face is our Minkowski sum

$$K_0(\mathcal{H}_{C,F})(\text{Pol}(X)) + \sum_{\gamma \in \Gamma \cap \overline{F}} [0, |\lambda_\gamma|\delta].$$

When equipped with adequate partitions, this can be regarded as a MV polytope of type  $\tilde{A}_1 \times \tilde{A}_0^{r-1}$ .

We now need to look at these partitions. In particular, we need to show that the partitions used in Theorem 7.24 (including those that decorate the polytope  $\text{Pol}(X)$ ) are the same as the partitions provided by Theorem 7.19, which decorate  $\text{Pol}(T)$ .

## 7.6 The MV polytope of a component

For any spherical Weyl chamber  $C$ , we have bijections

$$\mathfrak{I}_C \times \mathfrak{R}_C \times \mathfrak{P} \rightarrow \mathfrak{B} \quad \text{and} \quad \mathcal{P}^{\Gamma \cap \overline{C}} \rightarrow \mathfrak{R}_C,$$

by Proposition 4.6 and Theorem 7.19. Therefore each  $\Lambda_b \in \mathfrak{B}$  provides a tuple of partitions  $(\lambda_\gamma)_{\gamma \in \Gamma \cap \overline{C}}$ .

Concretely, this means that for any  $\eta \in C$  and any general point  $T$  in  $\Lambda_b$ , in the Krull-Schmidt decomposition of  $T_\eta^{\max}/T_\eta^{\min}$ , there are as many  $\gamma$ -cores of dimension-vector  $n\delta$  as parts equal to  $n$  in  $\lambda_\gamma$ .

A priori,  $\lambda_\gamma$  depends on  $b$ , on  $\gamma$  and on  $C$ . We suppress the dependence on  $b$  by fixing the latter, and we indicate the dependence on  $C$  in the notation by writing  $\lambda_\gamma(C)$ . Our aim now is to show that  $\lambda_\gamma(C)$  is in fact independent of  $C$ .

For this, we study what happens around a facet  $F$ . So let us consider a flag  $(C', F)$ , whence a functor  $\mathcal{H}_{C', F}$  to which we apply Theorem 7.24. Tracing  $\Lambda_b \in \mathfrak{B}$  through the bijections

$$\mathfrak{J}_F \times \mathfrak{R}_F \times \mathfrak{P}_F \rightarrow \mathfrak{B} \quad \text{and} \quad \mathfrak{A} \times \mathcal{P}^{\Gamma \cap \overline{F}} \rightarrow \mathfrak{R}_F$$

given by Proposition 4.6 and Theorem 7.24, we get  $(Z, (\lambda_\gamma(F))) \in \mathfrak{A} \times \mathcal{P}^{\Gamma \cap \overline{F}}$ . Concretely, for any  $\theta \in F$  and any general point  $T$  in  $\Lambda_b$ , the module  $T_\theta^{\max}/T_\theta^{\min}$  is the direct sum of  $\mathcal{H}(V)$ , where  $V$  is a general point of  $Z$ , and of indecomposable  $\gamma$ -cores of dimension-vector  $n\delta$ , with as many copies as parts equal to  $n$  in  $\lambda_\gamma(F)$ , for all  $n \geq 1$  and all  $\gamma \in \Gamma \cap \overline{F}$ . In addition, with the notation of Section 7.4, we can look at the  $\Pi$ -module  $V_{\gamma'}^{\max}/V_{\gamma'}^{\min}$ ; this is the general point of an irreducible component  $I(\lambda')$ . Likewise, the  $\Pi$ -module  $V_{\gamma''}^{\max}/V_{\gamma''}^{\min}$  is the general point of an irreducible component  $I(\lambda'')^*$ . Lastly, let  $C''$  be the other spherical Weyl chamber bounded by  $F$ .

**Lemma 7.25** *In the context above,*

$$\lambda_{\gamma_{C'/F}}(C') = \lambda', \quad \lambda_{\gamma_{C''/F}}(C'') = \lambda'',$$

and for each  $\gamma \in \Gamma \cap \overline{F}$ ,

$$\lambda_\gamma(C') = \lambda_\gamma(C'') = \lambda_\gamma(F).$$

*Proof.* Take  $m$  large enough and consider  $\eta = m\theta + \gamma_{C'/F}$ . This is an element of  $C'$ , so the module  $T_\eta^{\max}/T_\eta^{\min}$  bears the information about the partitions  $\lambda_\gamma(C')$ . Proposition 3.4 explains how this module can be obtained from  $T_\theta^{\max}/T_\theta^{\min}$ . The summands that are  $\gamma$ -cores, with  $\gamma \in \Gamma \cap \overline{F}$ , stay unchanged in the process, because they belong to  $\mathcal{R}_{C'}$ .

For its part however,  $\mathcal{H}(V)$  is truncated to its subquotient  $X' = \mathcal{H}(V_{\gamma'}^{\max}/V_{\gamma'}^{\min})$ . (One can check the equality  $\gamma' = \gamma_{C'/F} \circ K_0(\mathcal{H}_{C', F})$  with the help of (7.4).) Thus  $X'$  is the general point of the component  $\mathfrak{H}_{C', F}(I(\lambda')) = I(\gamma_{C'/F}, \lambda')$ .

This reasoning shows that

$$\lambda_\gamma(C') = \begin{cases} \lambda_\gamma(F) & \text{if } \gamma \in \Gamma \cap \overline{F}, \\ \lambda' & \text{if } \gamma = \gamma_{C'/F}. \end{cases}$$

The partitions  $\lambda_\gamma(C'')$  are computed in a similar fashion, using Corollary 7.22 at the last step.  $\square$

This lemma says in particular that  $\lambda_\gamma(C') = \lambda_\gamma(C'')$  if  $C'$  and  $C''$  are two adjacent spherical Weyl chambers. This implies that  $\lambda_\gamma(C)$  is independent of  $C$ , assuming of course that  $\gamma \in \overline{C}$ .

To an irreducible component  $\Lambda_b \in \mathfrak{B}$ , we may thus associate a family of partitions  $(\lambda_\gamma) \in \mathcal{P}^\Gamma$ . In addition, by Proposition 4.2 (ii),  $\Lambda_b$  contains a dense open subset on which the map  $T \mapsto \text{Pol}(T)$  is constant. As in the introduction, we denote by  $\widetilde{\text{Pol}}(b)$  the datum of this constant value  $\text{Pol}(T)$  and of the family of partitions  $(\lambda_\gamma)$ .

We have shown that  $\widetilde{\text{Pol}}(b)$  belongs to  $\mathcal{MV}$ :

- Its normal fan has been described in Corollary 7.3.
- Given a spherical Weyl chamber  $C$ , the partitions  $(\lambda_\gamma)$  for  $\gamma \in \overline{C}$  describe  $T_\theta^{\max}/T_\theta^{\min}$ , where  $T$  is general in  $\Lambda_b$  and  $\theta \in C$ . Therefore

$$\left( \sum_{\gamma \in \Gamma \cap \overline{C}} |\lambda_\gamma| \right) \delta = \underline{\dim} T_\theta^{\max}/T_\theta^{\min}.$$

- The shape of the non-vertical 2-faces of  $\text{Pol}(T)$  are constrained by the relations described in Propositions 5.27 and 5.28.
- Each vertical 2-face of  $\widetilde{\text{Pol}}(b)$  is an MV polytope of type  $\tilde{A}_1 \times \tilde{A}_0^{r-1}$ , as explained at the end of Section 7.5 and by Lemma 7.25.

At this point, we have proved every claim we made in the introduction, except Theorems 1.5 and 1.6.

## 7.7 Lusztig data (proof of Theorem 1.6)

In Section 7.2, we have associated a torsion pair  $(\mathcal{T}(A), \mathcal{F}(A))$  to each biconvex subset  $A \subseteq \Phi_+$ . In addition, we have seen in Section 2.4 that a convex order  $\preccurlyeq$  on  $\Phi_+$  amounts to the datum of a maximal totally ordered subset  $\mathcal{U}(\preccurlyeq)$  of  $\mathcal{V}$ , the set of all biconvex subsets. Given  $\preccurlyeq$ , we thus have a nested family of torsion pairs  $(\mathcal{T}(A), \mathcal{F}(A))_{A \in \mathcal{U}(\preccurlyeq)}$ , which endows each  $\Lambda$ -module  $T$  with a filtration. All these torsion pairs satisfy the openness condition (O) from Section 4.5, hence go down to the crystal  $\mathfrak{B}$ . By Proposition 4.6, the whole nested family goes down to the crystal. This procedure allows to write the set  $\mathfrak{B}$  as a product of simpler pieces. The aim of this section is to make this idea precise.

Fix a convex order  $\preccurlyeq$  and a weight  $\nu \in \mathbb{N}I$ . We set  $E = \{\alpha \in \Phi_+^{\text{re}} \mid \text{ht } \alpha \leq \text{ht } \nu\}$ . Enumerate the elements in  $E \cup \{\delta\}$  in decreasing order:  $\beta_1 \succ \beta_2 \succ \cdots \succ \beta_\ell$ . For  $1 \leq k \leq \ell$ , set

$A_k = \{\alpha \in \Phi_+ \mid \alpha \succ \beta_k\}$  and  $B_k = \{\alpha \in \Phi_+ \mid \alpha \succcurlyeq \beta_k\}$ . These biconvex subsets provide a nested family of torsion pairs (here we write only the torsion classes):

$$\{0\} \subseteq \mathcal{T}(A_1) \subseteq \mathcal{T}(B_1) \subseteq \mathcal{T}(A_2) \subseteq \cdots \subseteq T(A_\ell) \subseteq \mathcal{T}(B_\ell) \subseteq \Lambda\text{-mod}.$$

On a  $\Lambda$ -module  $T$ , this induces a filtration

$$0 \subseteq T_1 \subseteq \overline{T}_1 \subseteq T_2 \subseteq \cdots \subseteq T_\ell \subseteq \overline{T}_\ell \subseteq T, \quad (7.6)$$

with  $\overline{T}_k/T_k \in \mathcal{F}(A_k) \cap \mathcal{T}(B_k)$ . If  $\dim T = \nu$ , then, by Remark 7.4, the only jumps in the filtration (7.6) occur between  $T_k$  and  $\overline{T}_k$ . If moreover  $T$  is a general point in an irreducible component  $Z \in \mathfrak{B}(\nu)$ , then each subquotient  $\overline{T}_k/T_k$  will be a general point in an irreducible component  $Z_k \in \mathfrak{F}(A_k) \cap \mathfrak{T}(B_k)$ , by Proposition 4.6. We thus get a bijection

$$\mathfrak{B}(\nu) \rightarrow \left\{ (Z_1, \dots, Z_\ell) \in \prod_{k=1}^{\ell} \left( \mathfrak{F}(A_k) \cap \mathfrak{T}(B_k) \right) \left| \sum_{k=1}^{\ell} \text{wt } Z_k = \nu \right. \right\}. \quad (7.7)$$

By Proposition 7.6, if  $\beta_k$  is real, then  $\mathcal{F}(A_k) \cap \mathcal{T}(B_k) = \text{add } L(A_k, B_k)$ , where  $L(A_k, B_k)$  is a rigid  $\Lambda$ -module of dimension-vector  $\beta_k$ . Therefore  $\mathfrak{F}(A_k) \cap \mathfrak{T}(B_k)$  is in one-to-one correspondence with  $\mathbb{N}$ : to a natural number  $n$  corresponds the closure in  $\Lambda(n\beta_k)$  of the orbit that represents  $L(A_k, B_k)^{\oplus n}$ .

On the other hand, if  $\beta_k = \delta$ , then  $\mathcal{F}(A_k) \cap \mathcal{T}(B_k)$  is the category  $\mathcal{R}_C$ , where  $C$  is the spherical Weyl chamber such that  $(A_k, B_k) = (A_\theta^{\min}, A_\theta^{\max})$  for  $\theta \in C$  (see Lemma 2.8 (i)). Then Theorem 7.19 provides a bijection between  $\mathfrak{R}_C$  and  $\mathcal{P}^{\Gamma \cap \overline{C}}$ .

The bijection (7.7) can thus be rewritten as

$$\mathfrak{B}(\nu) \rightarrow \left\{ ((n_\beta), (\lambda_\gamma)) \in \mathbb{N}^E \times \mathcal{P}^{\Gamma \cap \overline{C}} \left| \sum_{\beta \in E} n_\beta \beta + \left( \sum_{\gamma \in \Gamma \cap \overline{C}} |\lambda_\gamma| \right) \delta = \nu \right. \right\}.$$

Letting  $\nu$  run over  $\mathbb{N}I$  and assembling the resulting bijections, we get a bijection

$$\Omega_{\preccurlyeq} : \mathfrak{B} \rightarrow \mathbb{N}^{(\Phi_+^{\text{re}})} \times \mathcal{P}^{\Gamma \cap \overline{C}}.$$

This construction proves Theorem 1.6.

The reader may here observe that the map  $\Omega_{\mathbf{i}}$  constructed in Section 5.5 gives the beginning of  $\Omega_{\preccurlyeq}$  when the smallest roots for  $\preccurlyeq$  are, in order

$$\alpha_{i_1}, s_{i_1} \alpha_{i_2}, s_{i_1} s_{i_2} \alpha_{i_3}, \dots$$

In addition, Remark 5.26 (ii) and Propositions 5.27 and 5.28 justify referring to the bijection  $\Omega_{\mathbf{i}}$  as the partial Lusztig datum in direction  $\mathbf{i}$ . We are thus led to regard  $\Omega_{\preccurlyeq}$  as the Lusztig datum in direction  $\preccurlyeq$ . A further justification of this terminology is the fact that the components of  $\Omega_{\preccurlyeq}(b)$  can be read as the lengths and the decorations of the edges of the path in the 1-skeleton of  $\text{Pol}(b)$  defined by  $\preccurlyeq$ .

## 7.8 Proof of Theorem 1.5

*Proof of the injectivity of  $\widetilde{\text{Pol}}$ .* We choose a convex order  $\preccurlyeq$ . Let  $C$  be the spherical Weyl chamber such that (1.1) holds for  $\theta \in C$ .

An element  $\tilde{P} \in \mathcal{MV}$  is the datum of a lattice convex polytope  $P$  whose normal fan is a coarsening of the affine Weyl fan and of a family of partitions  $(\lambda_\gamma) \in \mathcal{P}^\Gamma$ , subject to certain conditions. To  $P$  and  $\preccurlyeq$ , the construction in Section 2.5 associates a collection of numbers  $(n_\alpha) \in \mathbb{N}^{(\Phi_+^{\text{re}})}$ . Adding to this the partitions  $\lambda_\gamma$  with  $\gamma \in \Gamma \cap \overline{C}$ , we get an element  $\widetilde{\Omega}_{\preccurlyeq}(\tilde{P}) \in \mathbb{N}^{(\Phi_+)} \times \mathcal{P}^{\Gamma \cap \overline{C}}$ . Moreover, the definitions have been chosen in such a way that the diagram

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\widetilde{\text{Pol}}} & \mathcal{MV} \\ \Omega_{\preccurlyeq} \searrow & & \swarrow \widetilde{\Omega}_{\preccurlyeq} \\ & \mathbb{N}^{(\Phi_+)} \times \mathcal{P}^{\Gamma \cap \overline{C}} & \end{array}$$

commutes. (The key point here is that the definition of the torsion pairs in Section 7.2 matches the construction of Section 2.5, as showed in Proposition 7.5.) The injectivity of  $\widetilde{\text{Pol}}$  then follows from the injectivity of  $\Omega_{\preccurlyeq}$ .  $\square$

*Proof of the surjectivity of  $\widetilde{\text{Pol}}$ .* Since  $\Omega_{\preccurlyeq}$  is surjective, it suffices to establish the injectivity of  $\widetilde{\Omega}_{\preccurlyeq}$ . We can choose any convex order  $\preccurlyeq$ , at our convenience. For example, we can assume that  $\preccurlyeq$  comes from a linear form  $\theta \in (\mathbb{R}I)^*$ , as explained in Example 2.11 (ii).

Let us fix  $((n_\alpha), (\lambda_\gamma)) \in \mathbb{N}^{(\Phi_+^{\text{re}})} \times \mathcal{P}^{\Gamma \cap \overline{C}}$ . We want to show that this pair is the Lusztig datum of at most one MV polytope  $\tilde{P}$ . With the notation of Section 2.5, we need to show that the datum of the partitions  $\lambda_\gamma$ , where  $\gamma \in \Gamma \cap \overline{C}$ , and of the numbers

$$\mu(A) = \begin{cases} \sum_{\alpha \in A} n_\alpha \alpha & \text{if } \delta \notin A, \\ n_\delta \delta + \sum_{\alpha \in A \cap \Phi_+^{\text{re}}} n_\alpha \alpha & \text{if } \delta \in A, \end{cases}$$



where  $A$  is a terminal section of  $\preceq$ , determine the data  $\lambda_\gamma$  and the  $\mu(A)$  for  $\tilde{P}$ , for all remaining chamber coweights  $\gamma$  and biconvex subsets  $A$ . Let us set

$$\nu = \sum_{\alpha \in \Phi_+^{\text{re}}} n_\alpha \alpha + \left( \sum_{\gamma \in \Gamma \cap \overline{C}} |\lambda_\gamma| \right) \delta.$$

Let us take a biconvex subset  $A$ . Assuming that  $A$  is finite or cofinite (which is sufficient for our purpose), we can find  $\eta_1 \in (\mathbb{R}I)^*$  such that  $A = \{\alpha \in \Phi^+ \mid \langle \eta_1, \alpha \rangle > 0\}$ . We can ask that  $\eta_1$  is general enough so as to determine a convex order, as explained in Example 2.11 (ii). This condition means that  $\eta_1$  avoids the countably many hyperplanes

$$H_{\alpha, \beta} = \{\zeta \in (\mathbb{R}I)^* \mid \zeta(\alpha)/\text{ht}(\alpha) = \zeta(\beta)/\text{ht}(\beta)\},$$

where  $(\alpha, \beta) \in \Phi_+^2$ . We can then find a piecewise linear path  $\eta_t$  in  $(\mathbb{R}I)^*$ , for  $t \in [0, 1]$ , that connects  $\theta = \eta_0$  to  $\eta_1$ , and that crosses the hyperplanes  $H_{\alpha, \beta}$  one at a time.

When this path crosses  $H_{\alpha, \beta}$ , the convex order defined by  $\eta_t$  mostly remains unchanged, except for the roots in the rank 2 subsystem spanned by  $\alpha$  and  $\beta$ , whose order is reversed (see Example 2.11 (i) for a description of the possible orders for a subsystem of type  $\tilde{A}_1$ ). At this precise moment, the Lusztig datum undergoes the change imposed by the condition on the 2-faces. Moreover, except for a finite number of crossings,  $\alpha$  or  $\beta$  will have a height larger than  $\text{ht } \nu$ ; then the corresponding Lusztig datum is zero, and there is no actual change. The Lusztig datum of  $\tilde{P}$  relative to the convex order defined by  $\eta_1$ , and hence the weight  $\mu(A)$ , can therefore be determined from  $((n_\alpha), (\lambda_\gamma))$ , as claimed.

A similar reasoning shows that all the partitions  $\lambda_\gamma$  part of the datum of  $\tilde{P}$  are determined by our  $((n_\alpha), (\lambda_\gamma)) \in \mathbb{N}^{(\Phi_+^{\text{re}})} \times \mathcal{P}^{\Gamma \cap \overline{C}}$ . All in all,  $\widetilde{\Omega}_\preceq$  is injective.  $\square$

## Appendix: Restriction to the tame quiver

The path algebra  $KQ$  of an acyclic quiver  $Q$  can be seen as a subalgebra of the completed preprojective algebra  $\Lambda_Q$  of  $Q$ . In our present situation of an extended Dynkin diagram,  $Q$  is tame, so its representation theory is very well understood, thanks to the work of Dlab and Ringel. In this appendix, we discuss our constructions in terms of the representation theory of  $Q$ .

We begin with a refinement to Theorem 7.9 in the case where  $F$  is a minimal face, that is, the ray generated by a chamber coweight  $\gamma$ . For  $(\mu, \nu) \in (\mathbb{Z}I)^2$ , we write  $\mu \geq \nu$  if  $\mu - \nu \in \text{NI}$ .

**Proposition A.1** *Let  $F = \mathbb{R}_{>0}\gamma$  be a minimal face in the spherical Weyl fan, and let  $L$  be a linkage class of simple objects in  $\mathcal{R}_\gamma$ . Then  $\sum_{S \in L} \underline{\dim} S \leq \delta$ .*

*Proof.* Theorem 7.9 distinguishes two kinds of simple objects, described in its assertions (i) and (ii). For the objects of the first kind, the desired property is proved in Corollary 7.23. In the sequel of this proof, we consider the other case, when the dimension-vectors of objects in  $L$  belong to  $\iota(\Phi^s)$ .

Choose a zero node in  $I$  and we set  $I_0 = I \setminus \{0\}$ . The spherical root system  $\Phi^s$  is then endowed with a basis, namely  $\{\pi(\alpha_i) \mid i \in I_0\}$ , whence a positive system  $\Phi_+^s$ , and a dominant spherical Weyl chamber  $C_0^s$ . As in Section 7.5, we identify the spherical Weyl group  $W_0$  with the parabolic subgroup  $\langle s_i \mid i \in I_0 \rangle$  of  $W$ .

First consider the case where  $F \subseteq \overline{C_0^s}$ . Then  $\gamma$  is a fundamental coweight  $\varpi_i$ , with  $i \in I_0$ .

Given a connected component  $J$  of  $I_0 \setminus \{i\}$ , we can look at the root system  $\Phi_J = \Phi \cap \mathbb{Z}J$ . This root system is finite and irreducible and comes with a natural basis, so it has a largest root  $\tilde{\alpha}_J$ . By (the dual of) [16], Lemma 2 (2), there is a unique  $\Lambda$ -module with socle  $S_0$  and dimension-vector  $\delta - \tilde{\alpha}_J$ ; we denote it by  $R_J$ .

We claim that the head of  $R_J$  is isomorphic to  $S_i$ . In fact,  $S_0$  does not occur in the head of  $R_J$ ; otherwise,  $S_0$  would be a direct factor of  $R_J$  (because it occurs in the socle of  $R_J$  and its Jordan-Hölder multiplicity in  $R_J$  is one), which is ruled out by the fact that  $R_J$  is indecomposable (the socle of  $R_J$  is simple) of dimension-vector  $\neq \alpha_0$ . If  $S_j$  occurs in the head of  $R_J$ , then we can produce a  $\Lambda$ -module  $X$  with socle  $S_0$  and dimension-vector  $\underline{\dim} R_J - \alpha_j$ ; the latter is then a root, by [16], Lemma 2 (1), and therefore  $\tilde{\alpha}_J + \alpha_j$  is a root; this forces  $j = i$ . If  $S_i$  occurred twice in the head of  $R_J$ , then  $\underline{\dim} R_J - 2\alpha_i$  would be a root, so  $\tilde{\alpha}_J + 2\alpha_i$  would be a root, which is incompatible with  $(\tilde{\alpha}_J, \alpha_i) = -1$ .

Next we claim that  $R_J$  is a simple object in  $\mathcal{R}_F$ . To prove that, it suffices to show that  $R_J$  is  $\varpi_i$ -stable, in other words, that  $\langle \varpi_i, \underline{\dim} R_J \rangle = 0$  and that  $\langle \varpi_i, \underline{\dim} (R_J/X) \rangle > 0$  for all nontrivial submodules  $X$  of  $R_J$ . The first equation comes from the fact that  $\Phi_J$  is contained in  $\ker \varpi_i$ . To prove the second equation, we observe that  $S_0$  is not a Jordan-Hölder component of  $R_J/X$  (because  $X$  contains the unique copy of  $S_0$  in  $R_J$ ), so the simple components in  $R_J/X$  are  $S_j$  with  $j \in I_0$ , and  $S_i$  appears at least once in  $R_J/X$ .

In addition, the modules  $S_j$ , for  $j \in I_0 \setminus \{i\}$ , are also simple objects in  $\mathcal{R}_F$ . We now claim that the modules  $R_J$  and  $S_j$  are all the simple objects in  $\mathcal{R}_F$  whose dimension-vectors are in  $\iota(\Phi^s)$ .

Indeed, let  $T \in \text{Irr } \mathcal{R}_F$  such that  $\underline{\dim} T \in \iota(\Phi_+^s)$ . The vector space  $T_0$  attached to the zero node is thus zero. The vector space  $T_i$  attached to  $i$  is then also zero, for  $\varpi_i(\underline{\dim} T) = 0$ . Thus  $T$  is an iterated extension of the modules  $S_j$  with  $j \in I_0 \setminus \{i\}$ . Since all these modules belong to  $\mathcal{R}_F$ , we conclude that  $T$  is one of these  $S_j$ .

On the other hand, let  $T \in \text{Irr } \mathcal{R}_F$  such that  $\beta = \underline{\dim} T$  belongs to  $\iota(\Phi_-^s)$ . The simplicity of  $T$  forbids any  $S_j$  with  $j \in I_0 \setminus \{i\}$  to appear in the socle or in the head of  $T$ . Since  $T \in \mathcal{R}_F$  and  $S_i \in \mathcal{S}_F$ ,  $S_i$  cannot appear in the socle of  $T$ . We conclude that  $\text{soc } T = S_0$ . This condition and  $\beta$  completely determine  $T$ , by [16], Lemma 2 (2). With the notations of [5], Section 3 (see also Example 5.13), we have  $T \cong N(\beta - \omega_0)$ . (To apply [5], Theorem 3.1, we note the existence of  $w \in W_0$  such that  $\beta = w\alpha_0$ , which implies  $\beta - \omega_0 = -ws_0\omega_0$ .) Equation (3.1) in [5] (or the proof of Lemma 2 in [16]) then says that

$$\dim \text{hd}_J T = \max(0, (\beta, \alpha_j) - \langle \omega_0, \alpha_j \rangle),$$

and we have seen that the left-hand side is zero for  $j \in I_0 \setminus \{i\}$ . Let us write  $\beta = \iota(-\alpha)$ , with  $\alpha \in \Phi_+^s$ . Then  $(\alpha_j, \alpha) \geq 0$  for each  $j \in I_0 \setminus \{i\}$ . In addition,  $\langle \varpi_i, \beta \rangle = 0$ , so the support of  $\alpha$  (a subset of  $I_0$ ) avoids the node  $i$ , and thus there is a connected component  $J$  of  $I_0 \setminus \{i\}$  such that  $\alpha \in \Phi_J^s$ . We conclude that  $\alpha = \tilde{\alpha}_J$ , and thus  $T = R_J$ .

We now claim that the simple objects linked to  $R_J$  are the  $S_j$  with  $j \in J$ . By [9], chapitre 6, §1, n° 6, Proposition 19, there is a sequence  $\beta_1, \dots, \beta_n$  of elements in  $\{\alpha_j \mid j \in J\}$  such that  $\beta_1 + \dots + \beta_k$  is a root for each  $k$  and  $\beta_1 + \dots + \beta_n = \tilde{\alpha}_J$ . Let  $N_{n+1} = R_J$ , and for  $1 < k \leq n$ , let  $N_k$  be the  $\Lambda$ -module with socle  $S_0$  and dimension-vector

$$\delta - (\beta_1 + \dots + \beta_{k-1}) = \delta - \tilde{\alpha}_J + (\beta_n + \dots + \beta_k).$$

Inspecting the proof of [16], Lemma 2 (2), we see that  $N_k$  is the middle term of a non-trivial extension of  $N_{k+1}$  by the module  $S_j$  with  $\beta_k = \alpha_j$ . This shows that  $S_j$  is linked to at least one of the simple components of  $N_{k+1}$ . This conclusion also holds for  $k = 1$ , since  $\text{Ext}_\Lambda^1(N_2, S_j) \neq 0$  by Crawley-Boevey's formula (4.2). To sum up, all the  $S_j$  with  $j \in J$  are linked to  $R_J$ .

Now take two different connected components  $J$  and  $K$  of  $I_0 \setminus \{i\}$ . By Schur's lemma,

$$\text{Hom}_\Lambda(S_j, S_k) = \text{Hom}_\Lambda(S_j, R_K) = \text{Hom}_\Lambda(R_J, S_k) = \text{Hom}_\Lambda(R_J, R_K) = 0$$

for any  $j \in J$  and  $k \in K$ . An easy calculation based on Crawley-Boevey's formula (4.2) then shows that the  $\text{Ext}^1$  between  $S_j$  or  $R_J$  and  $S_k$  or  $R_K$  is zero. So  $J$  and  $K$  give rise to different linkage classes.

To each connected component of  $I_0 \setminus \{i\}$  corresponds thus a linkage class, formed by  $R_J$  and the  $S_j$  with  $j \in J$ . The sum of the dimension-vectors of these objects is  $\delta - \tilde{\alpha}_J + \sum_{j \in J} \alpha_j$ , which is smaller than or equal to  $\delta$ , with equality if and only if  $J$  is of type A.

At this point, we have established the desired property in the case where  $F \subseteq \overline{C_0^s}$ . It remains to handle the case of a general face  $F$ . Let  $w \in W_0$  of minimal length such that  $w^{-1}F \subseteq \overline{C_0}$ . Then  $\gamma = w\varpi_i$  for a certain index  $i \in I_0$ , and  $w$  is  $I_0 \setminus \{i\}$ -reduced on the right. By Theorem 5.18

(and Lemma 2.2), we have equivalences of categories

$$\mathcal{R}_{\varpi_i} \xrightleftharpoons[\text{Hom}_\Lambda(I_w, ?)]{I_w \otimes_\Lambda ?} \mathcal{R}_F.$$

We can then transfer to  $\mathcal{R}_F$  the information obtained above for  $\mathcal{R}_{\varpi_i}$ .

What is at stake is the fact that the sum of the dimension-vectors of the simple objects in a linkage class is at most  $\delta$ . In  $\mathcal{R}_{\varpi_i}$ , this sum has the form  $\delta - \beta_J$ , where  $J$  is a connected component of  $I_0 \setminus \{i\}$  and  $\beta_J = \tilde{\alpha}_J - \sum_{j \in J} \alpha_j$ . Checking the classification of root systems, we observe that  $\beta_J$  is a root; using Lemma 2.2, we see that  $\beta_J \notin N_{w^{-1}}$ . Therefore  $w\beta_J$  is a positive root and  $w(\delta - \beta_J) = \delta - w\beta_J$  is less than or equal to  $\delta$ , as desired.  $\square$

Recall the framework of Section 4.1. The graph  $(I, E)$  can be endowed with several orientations  $\Omega$  (we only consider acyclic orientations). The datum of  $\Omega$  gives a quiver  $Q$ , whence an Euler form  $\langle \cdot, \cdot \rangle_Q$  on  $\mathbb{Z}I$ , defined as

$$\langle \lambda, \mu \rangle_Q = \sum_{i \in I} \lambda_i \mu_i - \sum_{a \in \Omega} \lambda_{s(a)} \mu_{t(a)}.$$

The symmetric bilinear form  $(\cdot, \cdot) : \mathbb{Z}I \times \mathbb{Z}I \rightarrow \mathbb{Z}$  is then the symmetrization of  $\langle \cdot, \cdot \rangle_Q$ .

The imaginary root  $\delta$  belongs to the kernel of  $(\cdot, \cdot)$ , so  $\langle \delta, ? \rangle_Q$  induces a linear form on  $\mathfrak{t}^*$ , in other words, an element  $\gamma_\Omega \in \mathfrak{t}$ . For example, in type  $\tilde{A}_1$ , there are two orientations

$$\Omega' : 0 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 1 \quad \text{and} \quad \Omega'' : 0 \begin{array}{c} \xleftarrow{\alpha^*} \\ \xrightarrow{\beta^*} \end{array} 1.$$

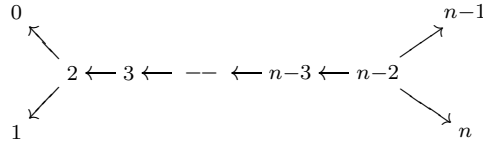
The corresponding linear forms are the chamber coweights  $\gamma_{\Omega'} = \gamma'$  and  $\gamma_{\Omega''} = \gamma''$  of Section 7.4.

**Proposition A.2** *The map  $\Omega \mapsto \gamma_\Omega$  is an injection from the set of all non-cyclic orientations of  $(I, E)$  into  $\Gamma$ . In type A, this map is bijective.*

*Sketch of proof.* We begin by studying the type  $\tilde{A}_n$ . The vertices of the graph  $(I, E)$  are numbered consecutively from 0 to  $n$  and we have  $\delta = \alpha_0 + \cdots + \alpha_n$ . Let  $\Omega$  be a non-cyclic orientation. The number  $a_i = \gamma_\Omega(\pi(\alpha_i)) = \langle \delta, \alpha_i \rangle_Q$  is equal to 1, 0 or  $-1$  depending on the number of arrows that terminate at  $i$ . When  $i$  cyclically runs over  $\{0, \dots, n\}$ ,  $a_i$  alternatively takes the values 1 and  $-1$ , with zeros interspersed between these values. The sum of all the  $a_i$  is  $\langle \delta, \delta \rangle_Q = 0$ , and the  $a_i$  cannot be all zero because  $\Omega$  is not cyclic. There is thus a unique sequence of values  $b_i \in \{0, 1\}$ , for  $i \in \{1, \dots, n+1\}$ , such that  $a_i = b_i - b_{i+1}$ , with the

convention  $b_0 = b_{n+1}$ . Now  $\mathfrak{t}^*$  has a standard realization as an hyperplane of the vector space with basis  $\{\varepsilon_i \mid 1 \leq i \leq n+1\}$ , to be specific, by writing  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ . The  $b_i$  thus appear as the coordinates of  $\gamma_\Omega$  in the basis  $(\varepsilon_i^*)$  dual to  $(\varepsilon_i)$ . In this basis, the chamber coweights are the sums  $\varepsilon_{i_1}^* + \cdots + \varepsilon_{i_k}^*$  with  $1 \leq k \leq n$  and  $1 \leq i_1 < \cdots < i_k \leq n+1$ . Certainly  $\gamma_\Omega$  matches this pattern, hence is a chamber coweight. We leave to the reader the routine verifications needed to show the announced bijectivity.

Consider the following orientation in type  $\tilde{D}_n$ .



A direct calculation shows that the associated coweight is  $(s_{n-1}s_n)(s_1 \cdots s_{n-2})\varpi_{n-2}$ . Since the graph is a tree, any orientation  $\Omega$  can be obtained from this one by a sequence of reflections at sources. Noting that  $\delta$  is  $W$ -invariant and using [5], Lemma 7.2, we deduce that the coweight  $\gamma_\Omega$  is  $W$ -conjugate to  $\varpi_{n-2}$ . We omit the proof of the injectivity of the map  $\Omega \mapsto \gamma_\Omega$ , for it requires lengthy (but direct) calculations in coordinates.

The types  $\tilde{E}_6$ ,  $\tilde{E}_7$  and  $\tilde{E}_8$  are dealt with similarly. One finds that the coweights  $\gamma_\Omega$  are all  $W$ -conjugate to the fundamental coweight corresponding to the branching point in the Dynkin diagram. For these exceptional types, we used a computer to check the injectivity of the map  $\Omega \mapsto \gamma_\Omega$ .  $\square$

Let us fix an orientation  $\Omega$ , whence a quiver  $Q$ . In [47], Ringel describes  $\Lambda$ -mod in terms of the category  $KQ$ -mod of finite dimensional representations of  $Q$ . More precisely, let  $\tau$  denote the Auslander-Reiten translation of  $KQ$ -mod and let  $M$  be a  $KQ$ -module. Then the structures of  $\Lambda$ -module on  $M$  that extend the given structure of  $KQ$ -module are in natural bijection with a certain subspace  $\mathcal{N}^{\tau^-}(M)$  of nilpotent elements in  $\mathcal{O}^{\tau^-}(M) = \text{Hom}_{KQ}(\tau^{-1}M, M)$ .

Recall that indecomposable  $KQ$ -modules are classified into preprojective, preinjective and regular types. Every  $KQ$ -module  $M$  can be written as  $M = I \oplus R \oplus P$ , where  $I$ ,  $R$  and  $P$  are the submodules of  $M$  obtained by gathering all direct summands in a Krull-Schmidt decomposition of  $M$  which are respectively preinjective, regular, and preprojective. (The subspaces  $R$  and  $P$  depend on the choice of the Krull-Schmidt decomposition, but  $I$  and  $I \oplus R$  do not; see [15], §7, Remark.)

**Proposition A.3** *Let  $T$  be a  $\Lambda$ -module and write a decomposition  $T|_Q = I \oplus R \oplus P$  as above. Then  $T_{\gamma_\Omega}^{\min} = I$  and  $T_{\gamma_\Omega}^{\max} = I \oplus R$  (as subspaces of  $T$ ).*

*Proof.* According to Ringel's construction, the datum of  $T$  is equivalent to the datum of  $M = T|_Q$  and of  $f \in \mathcal{N}^-(M)$ . By [15], §7, Lemma 3,  $f$  must map  $\tau^{-1}I$  to  $I$  and  $\tau^{-1}(R \oplus I)$  to  $R \oplus I$ , so  $I$  and  $R \oplus I$  are  $\Lambda$ -submodules of  $T$ .

Any nonzero quotient  $KQ$ -module of  $I$  is preinjective (otherwise, we would have a nonzero map from a preinjective to a preprojective or a regular module), hence has a positive defect (§7, Lemma 2 in [15]). A fortiori, a nonzero quotient  $\Lambda$ -module  $X$  of  $I$  satisfies  $\langle \gamma_\Omega, \underline{\dim} X \rangle > 0$ . Therefore the  $\Lambda$ -module  $I$  belongs to  $\mathcal{I}_{\gamma_\Omega}$ .

Similarly, a nonzero  $KQ$ -submodule of  $P \oplus R$  cannot have a preinjective direct summand, so has a nonpositive defect. Thus a nonzero  $\Lambda$ -submodule  $Y$  of  $T/I$  satisfies  $\langle \gamma_\Omega, \underline{\dim} Y \rangle \leq 0$ , and so  $T/I$  belongs to  $\overline{\mathcal{P}}_{\gamma_\Omega}$ .

We conclude that  $I$  is the torsion submodule of  $T$  with respect to the torsion pair  $(\mathcal{I}_{\gamma_\Omega}, \overline{\mathcal{P}}_{\gamma_\Omega})$ , so  $T_{\gamma_\Omega}^{\min} = I$ . The proof of the equality  $T_{\gamma_\Omega}^{\max} = I \oplus R$  is similar.  $\square$

*Remarks A.4.* (i) This proposition explains our choice of the notation  $\mathcal{I}_\theta$ ,  $\mathcal{R}_\theta$  and  $\mathcal{P}_\theta$ : when  $\theta = \gamma_\Omega$ , the objects of these categories are the  $\Lambda$ -modules whose restriction to  $Q$  are preinjective, regular or preprojective, respectively.

- (ii) The abelian category  $\mathcal{R}_Q$  of regular  $KQ$ -modules is well-understood (see [15], §8). Indecomposable objects are grouped into tubes, and there is no nonzero morphism or extension between modules that belong to different tubes. Simple objects in  $\mathcal{R}_Q$  lie at the mouth of the tubes; two simple objects are linked if and only if they belong to the same tube. The sum of the dimension-vectors of the simple objects in a tube  $\mathcal{T}$  is equal to  $\delta$ . A tube is called homogeneous if it has only one simple object; all but at most three tubes are homogeneous.

This description fits well with Theorem 7.9 and Proposition A.1, with  $F = \mathbb{R}_{>0}\gamma_\Omega$ . In fact, using Ringel's description, one easily shows that the map  $T \mapsto T|_Q$  is a bijection from  $\text{Irr } \mathcal{R}_F$  onto  $\text{Irr } \mathcal{R}_Q$ . (More precisely, if  $T \in \text{Irr } \mathcal{R}_F$ , then the arrows in  $\Omega^*$  act by zero on  $T$ .) In this context, the statements (i) and (ii) in Theorem 7.9 correspond to the cases where  $T|_Q$  belongs to an homogeneous tube or not.

Two simple objects in  $\mathcal{R}_F$  are linked if their restrictions to  $Q$  are linked in  $\text{Irr } \mathcal{R}_Q$ . Using Proposition A.1, we conclude that the bijection  $T \mapsto T|_Q$  maps linkage classes in  $\text{Irr } \mathcal{R}_F$  to linkage classes in  $\text{Irr } \mathcal{R}_Q$ .

- (iii) The last argument in (ii) implies that  $\sum_{S \in L} \underline{\dim} S = \delta$  holds for each linkage class  $L$  in  $\mathcal{R}_F$ . Let us adopt the notation of the proof of Proposition A.1: we choose a zero node in  $I$  and let  $i \in I_0$  be such that  $\gamma_\Omega$  is  $W_0$ -conjugated to  $\varpi_i$ . During the course of this proof, we observed that the condition on the sum of the dimension-vectors means that

the connected components  $J$  of  $I_0 \setminus \{i\}$  are of type  $A$ . This forces  $i$  to be the central node of  $I_0$  when  $I$  is of type  $\tilde{D}$  or  $\tilde{E}$ , a phenomenon already noticed during the proof of Proposition A.2.

Let  $\nu \in \mathbb{N}I$  be a dimension-vector. The nilpotent variety  $\Lambda(\nu)$  is a subvariety of the representation space of the double quiver  $\overline{Q}$ , which itself can be identified with the cotangent of the representation space  $\text{Rep}(KQ, \nu)$  of the quiver  $Q$ . It turns out that any irreducible component of  $\Lambda(\nu)$  can be written as  $\overline{T_X^*}$ , the closure of the conormal of a constructible subset  $X \subseteq \text{Rep}(KQ, \nu)$ . The relevant subsets  $X$  were first described by Lusztig [40] in the case of a bipartite orientation  $\Omega$ , and by Ringel [48] in the general case of an acyclic orientation. We now explain how this works.

Recall that an indecomposable  $KQ$ -module  $N$  is regular if and only if the Auslander-Reiten translation acts periodically on  $N$ : there is a number  $p > 0$  such that  $\tau^p N \cong N$ . One says that a finite-dimensional  $KQ$ -module is aperiodic if it does not contain a direct summand isomorphic to a module of the form  $\bigoplus_{i=0}^{p-1} \tau^i N$ , where  $N$  is an indecomposable regular module of  $\tau$ -period  $p$ . In addition, recall that a homogeneous tube  $\mathcal{T}$  contains exactly one module in each dimension-vector  $n\delta$ ; we denote this module by  $J(\mathcal{T}, n)$ . Lastly, recall that the set of homogeneous tubes is parametrized by the projective line  $\mathbb{P}_K^1$ , minus at most three points.

Given  $\nu \in \mathbb{N}I$ , let  $\mathcal{S}(\nu)$  be the set of all pairs  $(\sigma, \lambda)$ , where  $\sigma$  is an isomorphism class of aperiodic modules and  $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell)$  is a partition, with the further condition  $\nu = \underline{\dim} \sigma + |\lambda|\delta$  (see [40], §4.13). For  $(\sigma, \lambda) \in \mathcal{S}(\nu)$ , let  $X(\sigma, \lambda)$  be the set of all points in  $\text{Rep}(KQ, \nu)$  isomorphic to a module of the form  $M \oplus J(\mathcal{T}_1, \lambda_1) \oplus \dots \oplus J(\mathcal{T}_\ell, \lambda_\ell)$ , where  $\mathcal{T}_1, \dots, \mathcal{T}_\ell$  are distinct homogeneous tubes. Let also  $\mathcal{N}(\sigma, \lambda)$  be the conormal bundle of the closure of  $X(\sigma, \lambda)$ . Proposition 4.14 in [40] claims that  $\mathcal{N}$  is a bijection from  $\mathcal{S}(\nu)$  onto  $\mathfrak{B}(\nu)$ . Thanks to [48], Corollary 5.3,  $\mathcal{N}(\sigma, \lambda)$  can also be described as the closure of  $\{T \in \Lambda(\nu) \mid T|_Q \in X(\sigma, \lambda)\}$ .

With all these tools in hand, we can propose an exercise to the reader: prove that  $I(\gamma_\Omega, \lambda) = \mathcal{N}(0, \lambda)$  for each partition  $\lambda$ .

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